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Weak Consensus with Discrete-Valued Input and the Performance Dependency on Its Network Topology

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***Abstract*— This paper presents a new consensus algorithm for multi-agent systems with discrete-valued input. The algorithm is designed with consideration that the convergence speed does not degrade. The relation between the performance of the algorithm and the network topology of the system is analyzed.**

I. INTRODUCTION

Multi-agent systems are composed of many small systems called "agents" which cooperate each other and achieve common tasks [1]–[4]. In many cases, agents cooperate by communication and share their states each other. If the systems become large scale, the communication traffic will also become high. Therefore, it is important to maintain each traffic low and keep the capability of a system scale.

In particular, we consider a consensus problem, which is one of the basic problem in multi-agent systems.

There have also been some previous results for consensus problem under communication constraints [5]–[7]. But, this paper presents a new algorithm for consensus. The proposed algorithm is that an input of an agent are discrete-valued which is generated by state difference between the agent itself and its neighboring agents. The dynamics of the agents is given by the first ordered discrete-time model. By these settings, the consensus algorithm is composed by only discrete values. This fact will be advantage to implementation. This is because a lot of computers or other devices are needed to construct multi-agents systems and available cost for each device is limited. Thus, low cost digital devices are easily used when a discrete-valued consensus algorithm is applied.

In addition, the inputs are discretized not to be cause for degradation of the convergence speed. Conversely, the system does not achieve complete consensus with the proposed algorithm, that is, the states of the agents do not always become the same values at the steady state and it is depend on the network topology of the systems. Thus, the second purpose of this paper is clarifying the relation between the behavior at the steady state and the network topology of the multi-agent system.

II. PROPOSED CONSENSUS ALGORITHM

We consider the multi-agent system M with $n \in \mathbb{N}$ agents. The network topology is given by an undirected connected graph $G(V, E)$, where $V = \{1, 2, \ldots, n\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges. The set of neighboring agents of i is denoted by $\mathcal{N}_i$, so $j \in \mathcal{N}_i$ if $(i, j) \in E$.

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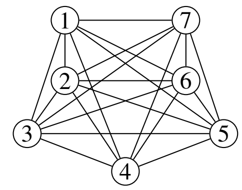

Fig. 1. Undirected complete graph.

The agent i updates its state $x_i(k) \in \mathbb{R}$ by

$$x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{N}} u_{ij}(k), \tag{1}$$

where $k \in \{0\} \cup \mathbb{N}$ is the discrete time and $u_{ij}(k) \in \mathbb{R}$ is the input from the neighboring agent $j \in \mathcal{N}_i$. The input $u_{ij}(k)$ is discrete-valued and given by

$$u_{ij}(k) = \begin{cases} b & \text{if } x_j(k) - x_i(k) > a, \\ -b & \text{if } x_j(k) - x_i(k) < -a, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$ are the constant values.

The condition of the input (2) means that if the state of the neighbors of agent i is far from the distance a, agent i updates its state to be the near value to the neighbors. The input (2) looks that the communication traffic does not decrease. However, the following situation is a good example where "communication traffic" is low. When we regard vehicles and their positions as the agents and their states, the communication of the systems correspond to "sensors which react by the distance a." In this case, the output of the sensors are binary and it can be implemented by low-cost digital devices.

Now we show how do multi-agent systems behave with this algorithm by three numerical examples. First, we consider the system whose network topology is given by a complete graph with $n = 7$ shown in Fig. 1. The initial value is given by $x_0 = x(0) = [-16\ -6\ -2\ 0\ 2\ 6\ 16]^\top$ and the constants in the input 2 are given by $a = 1,\ b = 1$. In this case, the behavior of the agents becomes Fig. 2 and oscillation in the steady state is found. Second, we change the initial value to $x_0 = [-12\ -6\ -2\ 4\ 2\ 9\ 12]^\top$ and keep the other conditions the same. The behavior becomes Fig. 3 and the states of all of agents converge to a constant value. Finally, we use another network topology shown in Fig. 4 When the initial value is given by $x_0 = [-16\ -6\ -2\ 0\ 2\ 6\ 16]^\top$ and $a = 1,\ b = 1$, the behavior of the agents becomes like Fig. 5. The response does not oscillate nor converge to

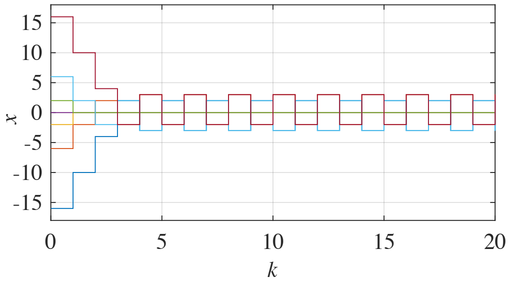

Fig. 2. Oscillated result with the complete graph.

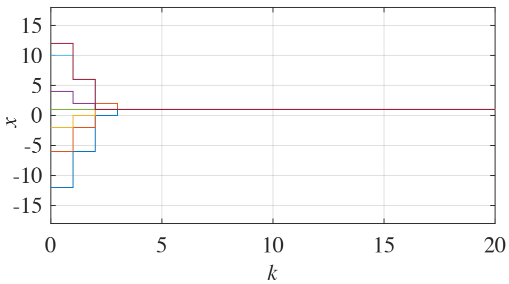

Fig. 3. Converged result with the complete graph.

a unique value. Each state of the agents converges to each value in contrast to the other two example.

From these example, we can find that the behavior of a system depends on its network topology and initial value. So, this paper focus on the network topology and give a solution for the system with which topology does not achieve consensus well.

III. PROBLEM FORMULATION

To evaluate how the systems achieve consensus, we introduce the performance index. We define the maximum distance $C(k) \in \mathbb{R}$ of the agents in the system at the time k as

$$C(k) = \max_{i,j} |x_j(k) - x_i(k)|. \tag{3}$$

For a time $\tau \in \mathbb{N}$, we assume that there exists $k^* > \tau$ which always satisfies $C(k^*) < C(\tau)$. Then, $\min_\tau C(\tau)$ means the maximum distance of the agents in the system at the steady state We formulate the problem by using this as follows:

Problem 1: For the multi-agent system M, the number of agents n, the initial value x_0, and the constant values a, b are given. When the network topology is given by an undirected connected graph, determine the maximum value of "the maximum distance of the agents in the system M" at the steady state, that is, determine the value of

$$\max_{x_0} \max_{G} \min_{\tau} C(\tau) \tag{4}$$

and find the graph G maximizing $\min_\tau C(\tau)$.

The equation (4) evaluates the worst case of consensus by the maximum distance of the states of the agents. Therefore, Problem 1 means which network topology does NOT achieve consensus well.

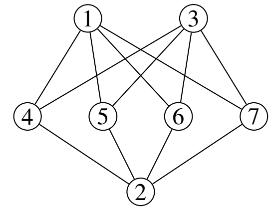

Fig. 4. Example of uncomplete graph.

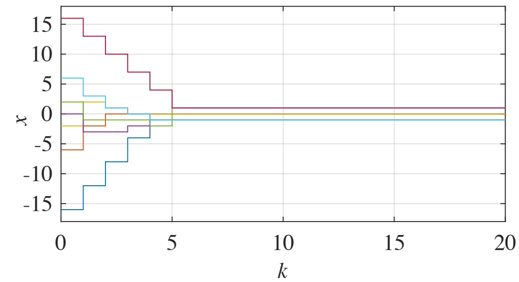

Fig. 5. Result with the graph of Fig.4 .

IV. MAIN RESULT

The following theorem is derived.

Theorem 1: For the multi-agent system M, the number of agents n, the initial value x_0, and the constant values a, b are given. When the network topology is given by an undirected connected graph, the maximum value of the maximum distance of the agent at the state steady is

$$\max_{x_0} \max_{G} \min_{\tau} C(\tau) = \begin{cases} 2a & \text{if } n \le \dfrac{a}{b} + 2, \\ 2(b(n-1) - (a+b)) & \text{otherwise.} \end{cases} \tag{5}$$

One of the graph maximizing $\min_\tau C(\tau)$ is that there are two nodes i^* which satisfy $(i^*, j) \in E,\ \forall j \in V \setminus \{i^*\}$ and the other nodes satisfy $(j_1, j_2) \notin E,\ (j_1 \neq j_2)$.

(Proof) We consider only the case $n > a/b + 2$. For a time k, we assume that there exists $\epsilon \in \mathbb{R}_+$ such that

$$\begin{aligned} C(k) &= \max_{ij}(x_j(k) - x_i(k)) \\ &= 2(b(n-1) - (a+b)) + \epsilon. \end{aligned} \tag{6}$$

Then, for a time $k' > k$ there exist a time τ, $(k < \tau \le k')$, which always satisfies

$$C(k') \le 2(b(n-1) - (a+b)),\ \forall k. \tag{7}$$

On the other hand, for any time $k' \ge \tau$, there also exists a combination of a graph G and the initial value x_0 where $C(k') = C(k'+2) = 2(b(n-1) - (a+b))$ is satisfied. □

The network topology in Theorem 1 is the graph with two nodes which have edges towards every agents except themselves. The graph are composed by two star topology intersecting each other like Fig.6.

Here we confirm Theorem 1 by the examples in Section II. In the examples, the conditions are $n = 7, a = 1, b = 1$. Therefore, the worst case of the maximum distance derived

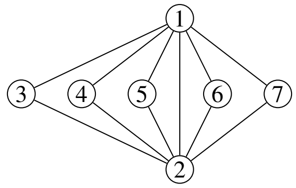

Fig. 6. One of the worst case of network topology.

from Theorem 1 is $\max_{x_0} \max_G \min_\tau C(\tau) = 8$. The worst result in three examples was the case of Fig. 3 and the maximum distance at the steady state was also 8. We have tried some other examples and they also satisfy Theorem 1.

V. CONCLUSIONS

In this paper, we have proposed a new consensus algorithm with discrete-valued input. We have clarified the relation between the performance and the networked topology and shown the worst case of topology. In the result, it have been clarified that the maximum distance of the states in the system at steady state can be maximized when the network has two star topology intersected each other. The analysis in this paper is just about the performance at the steady state. Quantitative evaluation in the transient state *e.g.,* convergence speed is still an open problem.

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