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Consensus-induced Centrality for Networks of Dynamical Systems

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Abstract— **In multi-agent systems where agents are coupled through the underlying networks and follow the consensus dynamics, the ability for agents on networks to reach consensus depends on the network structure, and more specifically depends on the spectra of the Laplacian of the networks. Each agent on the network plays a role in aggregating the information for the agent population to reach a consensus. To evaluate the influence that one agent would have on the network to reach consensus, we propose the consensus-induced centrality measure. Examples of consensus-induced centrality analysis for different types of networks are given. The consensus-induced centrality has possible applications in many network systems, such as social networks, power grid stability and wireless sensor networks, etc.**

Index Terms— **Systems on networks, centrality for network systems, multi-agent systems, induced centrality**

I. INTRODUCTION

Networks are everywhere ranging from social networks, biological networks to engineering networks. Underlying most networks, there are dynamical processes. To fully understand and control these networks, it is important to study both the network structure and the underlying dynamics. For instance, in the study of consensus and synchronization on networks, the dynamics is specified locally following consensus protocol [1], [2], [3] and the key structure property is represented by the spectrum of the graph Laplacian.

Most graph theoretic measures for node importance (centrality) on networks tend to focus on the structural importance of nodes in the underlying graph [4]. However, to study network dynamical systems, it is more meaningful to represent the node importance considering also the underlying dynamical process on the networks.

This paper provides a measure for nodes representing their importance in a network under a dynamical setting. The underlying methodology is to evaluate the impact of the removal of an agent on the ability of the network to reach consensus or synchronization. The importance (centrality) values of the agents are ranked according to their impact factors. Further, since the ability to reach consensus is an invariant of the network system represented by the spectrum of the underlying graph, the change in the graph spectrum is used to evaluate the change of the ability to reach consensus. We define the consensus-induced centrality for a network agent as the change of the underlying graph spectrum with the removal of that agent. See [4] for general induced centrality measures on graphs.

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For unweighted graphs, the centrality based on the change of algebraic connectivity with removal of nodes is called the vertex-deleted centrality measure and is studied in [5] where tight lower bound and upper bound are established. The results in our paper apply to general weighted graphs, and lower bounds and upper bounds based on local connection weights are proven. These results are straightforward to generalize to centrality measures for groups of nodes. Most importantly, we give a clear interpretation of the induced centrality measure in the context of multi-agent network systems.

In this work, the centrality measure for connected networks is considered, however, it is straightforward to generalize the results to disconnected networks.

A. Basic notations

Let $G_n = (V_n, E, W)$ denote an undirected weighted graph with the node set V_n where n is the size of the graph, i.e. $|V_n| = n$, the edge set E and the positive edge weight set W. Simple graphs (i.e. unweighted graphs with no self-loops and no multiple edges) are a special case where W only takes 0 and 1 as its elements. Let $A(G_n)$ represent the symmetric adjacency matrix of the graph G_n where the ij^{th} element of $A(G_n)$ is a_{ij} if there is an undirected connection between the i^{th} node and the j^{th} node with edge weight a_{ij} in the graph G_n and zero otherwise. Let $\mathbf{1} = [1, 1, ..., 1]^T \in R^n$. Let $L(G_n)$ be the Laplacian of graph G_n, i.e. $L(G_n) = \text{diag}(A(G_n)\mathbf{1}) - A(G_n)$. Let the (real) eigenvalues of $L(G_n)$ be ordered as

$$\lambda_1(G_n) \leq \lambda_2(G_n) \leq, ... \leq \lambda_n(G_n).$$

We use the word "network" to refer to a network system with the underlying graph and the underlying dynamics which in the present case is the consensus type dynamics.

II. CONSENSUS AND SYNCHRONIZATION ON NETWORKS

A. Consensus on Networks

Consider a networks with n agents each of which follows the local dynamics and protocol

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

$$u_i(t) = -\sum_{j \in \mathcal{N}_i} a_{ij}(x_i(t) - x_j(t)), \tag{2}$$

$$x_i(t), u_i(t) \in R, \quad i \in \{1, ..., n\}, \quad t \in [0, \infty),$$

where x_i, u_i and $\mathcal{N}_i$ represent respectively the state, the control input and the neighbourhood set of agent i on the network, and a_{ij} denotes the edge weight connecting agent i and agent j. Denote the weighted adjacency matrix by $A =$

$[a_{ij}]$ and denote the Laplacian matrix by L. The system is said to reach a consensus at time t if $x_1(t) = x_2(t)... = x_n(t)$.

The system dynamics can be written in the following compact form

$$\dot{X}(t) = -LX(t), \tag{3}$$

where $X(t) = [x_1(t), ..., x_n(t)]^T$. The solution of the system is given by

$$\begin{aligned} X(t) &= e^{-Lt}X_0 = Ve^{-\Lambda t}V^T X_0 \\ &= e^{-\lambda_1 t}(v_1^T X_0)v_1 + e^{-\lambda_2 t}(v_2^T X_0)v_2 \\ &\quad \cdots + e^{-\lambda_n t}(v_n^T X_0)v_n \end{aligned} \tag{4}$$

where $X(0) = X_0$ and

$$L = V\Lambda V^T = [v_1, \cdots, v_n]\text{diag}(\lambda_1, ..., \lambda_n)[v_1, \cdots, v_n]^T,$$

where $v_1, ..., v_n$ are the normalized eigenvectors of L that correspond to the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ respectively. Under the proposed local control protocol (2), the system will reach a consensus iff

$$0 = \lambda_1 < \lambda_2. \tag{5}$$

From (4) we can see that if $\lambda_1 = 0$, in the direction of v_1 the system trajectory will always have a length $v_1^T X_0$, and the convergence rate of the consensus problem is determined by the second smallest eigenvalue λ_2 of the Laplacian. See [6] for more details.

B. Synchronization on networks

A typical dynamical network model [7], [8] for synchronization analysis is as follows. The dynamical network contains N identical nodes which are diffusively and linearly coupled. On each node there exist a $n-$dimensional dynamical system.

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) - c\sum_{j=1}^{N} L_{ij}\Gamma\mathbf{x}_j, \quad i = 1, 2, ..., N \tag{6}$$

with $\mathbf{x}_i = (x_{i1}, x_{i2}, ..., x_{in})^T \in R^n$ as the state vector of node i, constant $c > 0$ as the coupling strength, and $\Gamma \in R^{n\times n}$ a constant matrix with only 0-1 elements as variables' coupling structure. $L_{ij} = L_{ji} = -1 (i \neq j)$ if node i and node j are connected, otherwise $L_{ij} = L_{ji} = 0 (i \neq j)$. $L_{ii} = d_i$ where d_i is the connection degree of node i. Note that L is the Laplacian of the underlying graph representing connection between agents. The objective is to (asymptotically) synchronize the networks, i.e. to achieve

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = ... = \mathbf{x}_N(t) = \mathbf{s}(t), \text{ as } t \to \infty \tag{7}$$

where $\mathbf{s}(t)$ is the solution to $\dot{\mathbf{s}}(t) = f(\mathbf{s}(t))$. Assume the underlying network is connected, that is, the adjacency matrix $[a_{ij}]$ is irreducible. This implies $0 = \lambda_1 < \lambda_2 \leq ... \leq \lambda_N$. The synchronization state is exponentially stable if

$$c \geq T\lambda_2^{-1} \tag{8}$$

where $T > 0$ is a constant determined by the dynamics of an isolated node and the state mapping structure Γ and $c > 0$ is the coupling strength [9], [8]. This shows that the second smallest eigenvalue of the Laplacian of the underlying graph determines the synchronizability of the dynamical network (6).

There are other variations of network consensus and synchronization problems. The key index to these problems is the spectrum of the Laplacian of the underlying networks. The first eigenvalue of the graph Laplacian is always zero. The second smallest eigenvalue of the graph Laplacian (also called algebraic connectivity), determines the rate of convergence of consensus and synchronization capability of the network.

C. Consensibility and Relative Consensibility of Network Systems

Consider a network system with an underlying connected graph G_n. The *consensibility* (ability to reach consensus) of the network system is defined as the second smallest eigenvalue of the underlying graph Laplacian, i.e. the algebraic connectivity of the underlying graph, denoted by $\lambda_2(G_n)$. The *relative consensibility* of the network system is defined as $\frac{\lambda_2(G_n)}{n}$ representing the ability of the network reaching consensus compared to the fully connected network on the same node set.

D. Properties of Algebraic Connectivity

As is known, the *edge connectivity* $e(G)$ of an undirected simple graph $G = (V, E)$ is the smallest cardinality of a subset $E_1 \subset E$ satisfying the property that $G_1 = (V, E\backslash E_1)$ is not connected. Similarly, the *vertex connectivity* $v(G)$ of $G = (V, E)$ is the smallest cardinality of a subset $V_1 \subset V$ having the property that the subgraph G_1 generated by removing V_1 from G is not connected. The *minimum degree* $\delta(G_n)$ of graph G_n is the minimum degree of its nodes.

Theorem 1 ([10])**.** Let $G_n (n \geq 2)$ be a non-complete simple graph with vertex connectivity $\nu(G_n)$, edge connectivity $e(G_n)$ and the minimum degree $\delta(G_n)$. Then

$$\lambda_2(G_n) \leq \nu(G_n) \leq e(G_n) \leq \delta(G_n).$$

Theorem 2 ([11])**.** Let $G_n (n \geq 2)$ be a connected simple graph with n vertices. Then

$$2(1 - \cos\frac{\pi}{n}) \leq \lambda_2(G_n) \leq n$$

with the left equality if and only if G_n is a path.

Theorem 3 ([10])**.** Let $G_n (n \geq 2)$ be a simple graph. Then $\lambda_2(G_n) > 0$ if and only if G_n is connected.

This result generalizes to undirected positively weighted graphs.

Theorem 4. For an undirected and positively weighted graph $G_n = (V, E, W), n \geq 2$, $\lambda_2(G_n) > 0$, if and only if G_n is connected.

For the purpose of convenience, we attach the proof in the appendix.

III. Consensus-induced Centrality Measure

The importance of each node of the network in the process to reach consensus and synchronization can be reflected by the change of the graph algebraic connectivity if that particular node is removed. Based on this intuition, we define the consensus-induced centrality measure for networks. There can be two definitions: absolute consensus-induced centrality (ACIC) measure and relative consensus-induced centrality (RCIC) measure.

A. Absolute Consensus-induced Centrality Measure

We define the absolute consensus-induced centrality measure in the following. The subgraph of $G_n = (V_n, E, W)$ generated by removing node $v \in V_n$ of G_n is denoted by $G_{n-1}^{(v)}$. For a node v in a graph G_n, the absolute consensus-induced centrality (ACIC) value is given by

$$ACIC_v = \lambda_2(G_n) - \lambda_2(G_{n-1}^{(v)}). \tag{9}$$

This value is the value drop of the second smallest eigenvalue of the graph Laplacian after removing node v in the G_n. It reflects the drop of the rate of reaching consensus on G_n.

B. Relative Consensus-induced Centrality Measure

As we know that for a complete graph of size $n > 1$, the eigenvalues of its Laplacian are $\{0, n, \cdots n\}$. Since the two graphs G_n and $G_{n-1}^{(v)}$ are of different sizes and the size of the network determines on the maximum value of eigenvalue that graph achieve, it is meaningful to compare the relative change of the eigenvalues. Therefore we define the relative consensus-induced centrality (RCIC) value for node v of graph G_n as follows:

$$RCIC_v = \frac{\lambda_2(G_n)}{n} - \frac{\lambda_2(G_{n-1}^{(v)})}{n-1}. \tag{10}$$

This value measures the change of the ratio between the algebraic connectivity and the maximum algebraic connectivity of graphs on the same node set by removing node v in the graph G_n.

According to the definition of RCIC, if the network is sparse and network size is large, RCIC can be very small.

RCIC and ACIC would represent the same property if the network is very large. But for small networks, RCIC and ACIC represent relatively different properties.

IV. Basic Results on Consensus-induced Centrality Measure

A. Upper Bound of ACIC

Based on the Cauchy Interlace Theorem and the Courant-Fischer Theorem, in the appendix we prove the following result on the change of eigenvalues of the graph Laplacian when removing a node from the graph.

Lemma 1. Consider a graph $G_n = (V_n, E, W)$ and its subgraph $G_{n-1}^{(v)}$ generated by removing node v from G_n. Then

$$\lambda_k(G_{n-1}^{(v)}) \geq \lambda_k(G_n) - \alpha_{\max}^v, \quad 1 \leq k \leq n-1,$$

where $\alpha_{\max}^v$ is maximum of the connection weights between v and all other nodes in G_n.

We can see that the maximum drop in the relative eigenvalues of the Laplacians when removing a node v from graph G_n is upper bounded by $\alpha_{\max}^v$, i.e. the maximum of the connection weights between v and all other nodes in G_n.

Based on this result and the definition of ACIC, we obtain an upper bound for the ACIC of the nodes.

Theorem 5. Consider a network $G_n = (V_n, E, W)$. For any $v \in V_n$

$$ACIC_v \leq \alpha_{\max}^v,$$

where $\alpha_{\max}^v$ is maximum of the connection weights between v and all other nodes in G_n.

Corollary 1. Consider a network $G_n = (V_n, E, W)$. Then

$$\forall v \in V_n, \quad ACIC_v \leq \alpha_{\max},$$

where $\alpha_{\max}$ is the maximum edge weight in G_n.

$\alpha_{\max}$ provides the upper bound for $ACIC$ and hence a robustness estimate for synchronization when the removal of one node is possible. Specifically, if $\lambda_1 - \alpha_{\max} > 0$, then the coupling strength

$$c \geq T(\lambda_1 - \alpha_{\max})^{-1},$$

based on (8), ensures synchronization under the removal of any agent in the network.

B. Lower Bound of ACIC

Lemma 2. Consider a graph $G_n = (V_n, E, W)$ and its subgraph $G_{n-1}^{(v)}$ generated by removing node v from G_n. Then

$$\lambda_k(G_{n-1}^{(v)}) \leq \lambda_{k+1}(G_n) - \alpha_{\min}^v \quad 1 \leq k \leq n-1,$$

where $\alpha_{\min}^v$ is minimum of the connection weights (including zero weights) between v and all other nodes in G_n.

Theorem 6. Consider a network $G_n = (V_n, E, W)$. Then, for any $v \in V_n$

$$ACIC_v \geq \alpha_{\min}^v + \lambda_2(G_n) - \lambda_3(G_n),$$

where $\alpha_{\min}^v$ is minimum of the connection weights between v and all other nodes in G_n.

Corollary 2. Consider a network $G_n = (V_n, E, W)$. Then

$$\forall v \in V_n, \quad ACIC_v \geq \alpha_{\min} + \lambda_2(G_n) - \lambda_3(G_n),$$

where $\alpha_{\min}$ is the minimum connection weight (including zero weights) over all nodes.

Corollary 3. Consider a network $G_n = (V_n, E, W)$. Then

$$\forall v \in V_n, \quad ACIC_v \geq \lambda_2(G_n) - \lambda_3(G_n).$$

V. Examples of Consensus-induced Centrality Measure for Networks

We calculate the ACIC and RCIC for some simple networks in Table I and real world networks. For simple networks, since the edge weight is either 1 or 0, $\alpha_{\max} = 1$ (if the graph is non-empty) and therefore $ACIC \leq 1$.

TABLE I
ALGEBRAIC CONNECTIVITY FOR SOME SIMPLE NETWORKS [12]

Graph G	Algebraic Connectivity
Complete Graph K_n	$\lambda_2(K_n) = n$
Path $P_n, (n > 1)$	$\lambda_2(P_n) = 2(1 - \cos\frac{\pi}{n})$
Cycle $C_n, (n > 2)$	$\lambda_2(C_n) = 2(1 - \cos\frac{2\pi}{n})$
Bipartite complete graph $K_{p,q}$, $(p \geq 1, q > 1)$	$\lambda_2(K_{p,q}) = \min\{q, p\}$
Star $K_{1,q}, (q > 1)$	$\lambda_2(K_{1,q}) = 1$

A. Complete Networks

The nodes on a complete graph (i.e. fully connected graph) are indifferent from one another, and they all have the same importance. For any node v in an complete graph C_n:

$$ACIC_v = n - (n-1) = 1$$
$$RCIC_v = \frac{n}{n} - \frac{n-1}{n-1} = 0$$

ACIC is 1 means that the algebraic connectivity decreases by 1 if any node is removed. RCIC is 0 implies that compared to the full capacity of the network, the relative consensibility of the network does not change when any node is removed.

B. Complete Bipartite Networks

For a complete bipartite network $K_{|P|,|Q|}$ with n nodes where P and Q are the two complete cliques. Suppose $|P| < |Q|$. For a node v_p in P,

$$ACIC_{v_p} = |P| - (|P| - 1) = 1,$$
$$RCIC_{v_p} = \frac{|P|}{n} - \frac{|P|-1}{n-1}.$$

For a node v_q in Q

$$ACIC_{v_q} = |P| - |P| = 0,$$
$$RCIC_{v_q} = \frac{|P|}{n} - \frac{|P|}{n-1} = \frac{-|P|}{n(n-1)}.$$

This implies that nodes in the smaller clique P are more important than nodes in the larger clique Q in terms of consensibility. The removal of $(|Q|-|P|)$ nodes in the larger clique Q does not affect the consensibility of the network.

Star graphs are special cases of complete bipartite graphs where the smaller complete clique P has only one node. For the center node v_c in the star network $K_{1,n-1}, (n > 3)$,

$$ACIC_{v_c} = 1, \quad RCIC_{v_c} = \frac{1}{n}.$$

For any leaf node v_l in a star network $K_{1,n-1}$

$$ACIC_{v_l} = 0, \quad RCIC_{v_l} = -\frac{1}{n(n-1)}.$$

This implies that the center node is more important than the leaf nodes in terms of consensability on a star network. $RCIC$ is negative for leaf nodes implies that the remove a leaf node can increase the network relative consensibility.

C. Path Networks

For a node v_b on the boundary of a path network

$$ACIC_{v_b} = 2(\cos\frac{\pi}{(n-1)} - \cos\frac{\pi}{n}),$$
$$RCIC_{v_b} = \frac{2(1-\cos\frac{\pi}{n})}{n} - \frac{2(1-\cos\frac{\pi}{(n-1)})}{n-1}$$

For an internal node v_i on a path network P_n

$$ACIC_{v_i} = 2(1 - \cos\frac{\pi}{n}) - 0 = 2(1 - \cos\frac{\pi}{n}),$$
$$RCIC_{v_i} = \frac{2(1-\cos\frac{\pi}{n})}{n} - 0 = \frac{2(1-\cos\frac{\pi}{n})}{n}$$

This suggests that on path networks, nodes in the middle is more important than nodes on the boundary in terms of the influence to consensibility.

D. Cycle Networks

For any node v on a cycle network C_n $(n > 2)$

$$ACIC_v = 2(\cos\frac{\pi}{n} - \cos\frac{2\pi}{n-1})$$
$$RCIC_v = \frac{2(1-\cos\frac{2\pi}{n})}{n} - \frac{2(1-\cos\frac{\pi}{n-1})}{n-1}$$

E. A Simple Network with Negative ACIC

Here is a network example on which some nodes have negative ACIC and RCIC.

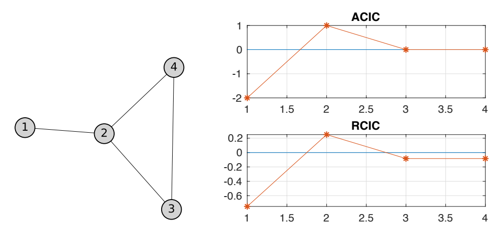

Fig. 1. A Simple Network with Negative ACIC

Negative ACIC and negative RCIC of node 1 respectively imply that its removal would increase the consensibility and relative consensibility of the network. Note that node 1 is weakly connected to the close community formed by nodes 2, 3 and 4. Node 2 which is located at the center of the network has the largest ACIC and the largest RCIC. Zero ACICs of nodes 3 and 4 suggest the removal of them does not change the absolute consensibility. Negative RCIC of nodes 3 and 4 implies that each removal would result in a network that has better relative consensibility.

F. The Karate Club Network

Here is the analysis on the Karate club network [13]. Node 1 has the largest ACIC and RCIC and the removal of which will disconnect the network. Node 34 has the second largest ACIC and RCIC. Nodes 2, 3, 32 and 33 has relatively large ACIC and RCIC compared to other nodes. Node 17 has the smallest negative ACIC and RCIC, which means its removal would increase network consensisibility the most. Nodes that have larger ACIC and RCIC tend to be the hubs of a densely

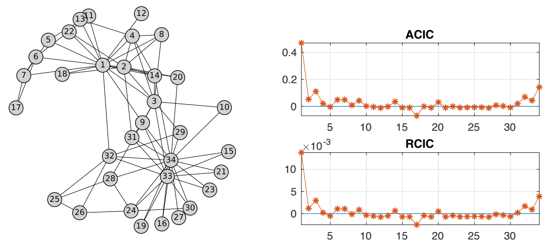

Fig. 2. The Karate Club Network

connected community and nodes that have small ACIC and RCIC tend to be nodes that are weakly connected to or distant from communities.

G. Small World Networks

We use the small world network model [14] to generate the following network, by starting with 20 nodes, connecting each node to 4 nearest neighbours in ring topology and for each edge the rewiring probability is 0.5. The node has the

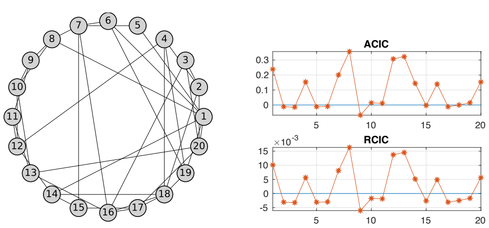

Fig. 3. Small World Network Example

highest ACIC and RCIC is node 8, and it has a degree of 4 which is not the highest node degree. Node 1 has the highest node degree which is 6. Node 9 has the lowest ACIC and RCIC and it has a node degree of 3.

H. Power Networks

An undirected, unweighted network with 4941 nodes representing the topology of the Western States Power Grid of the United States [14].

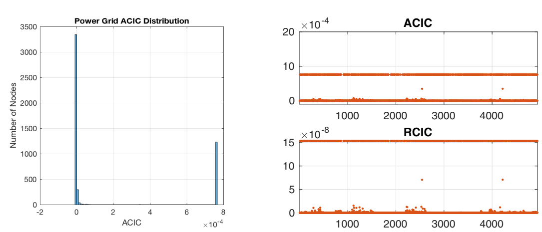

Fig. 4. The Power Network

The algebraic connectivity: 7.5921×10^{-4}. The max ACIC of all nodes is 7.5921×10^{-4}. There are 1229 nodes achieved the maximum ACIC under error bound 10^{-8}. The minimum ACIC of all nodes is -5.9562×10^{-7}. Under error bound 10^{-8}, there are 27 nodes that achieve the minimum ACIC.

Since the network size is large and the network is sparse, the RCIC would have very small values. The maximum RCIC is 1.5366×10^{-7}. The minimum RCIC is -1.5167×10^{-10}.

I. The Southern Women Club Network

The Southern Women Club network [15] is an undirected weighted network that contains the observed attendance at 14 social events by 18 southern women. The edge weight is the number of co-attended events. The maximum edge weight in the network is 7.

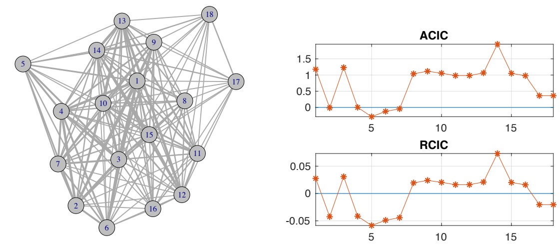

Fig. 5. The Southern Women Club Network

The network consensibility (algebraic connectivity) is 12.7572. Node 5 achieves the minimun ACIC (-0.2870) and minimum RCIC (-0.0586). Node 14 achieves the maximum ACIC (1.9528) and the maximum RCIC (0.0732).

VI. Discussions and Applications

An application is on power grid stability since the grid stability is interpreted as the synchronization of coupled oscillators [2]. If one node on the grid need to be removed, this work answers the question that which node can be remove first to ensure that the resulting network structure have at least the same performance in reaching synchronization.

In the case of wireless sensor networks where information need to be processed through aggregation, this can be used to reduce the use the redundant sensors to improve the consensus rate, by careful choice of nodes according to ACIC.

VII. Conclusion

In this work we propose and analyse the induced centrality measures for agents in network systems with consensus type dynamics. Future directions include: (1) consensus-induced measures for edges, (2) centrality measures for groups of nodes, (3) centrality measures for directed network systems, (4) other induced centrality measures for controlled dynamical network systems.

VIII. Acknowledgment

The authors would like to thank the anonymous reviewers their constructive comments on this paper.

Appendix

A. Proof of Theorem 4

Proof. Since the eigenvalues of $L(G_n)$ are non-negative, we only need to prove that $\lambda_2(G_n) = 0$ if and only if the graph G_n is not connected.

First, we prove that G_n is not connected implies $\lambda_2(G_n) = 0$. If G_n is not connected, it consists at least two separated subgraph, denoted by G^1 and G^2. Then the Laplacian $L(G_n)$ consists of two blocks $L(G^1)$ and $L(G^2)$, each of which has one zero eigenvalue. Therefore $L(G_n)$ has at least two zero eigenvalues and $\lambda_2(G_n) = 0$.

Second, we prove that $\lambda_2(G_n) = 0$ implies G_n is not connected. The algebraic connectivity of a weighted graph can be written as follows [16] :

$$\lambda_2(G_n) = \min_{x\neq 0, x\perp \mathbf{1}} \frac{\sum_{(i,j)\in E} a_{ij}(x_i - x_j)^2}{\sum_{i=1}^n {x_i}^2}, \tag{11}$$

where a_{ij} is the positive edge weight between node i and node j. $\lambda_2(G_n) = 0$ implies that there exist $x \neq 0$ such that $\sum_{i=1}^n x_i = 0$ and $\sum_{(i,j)\in E} a_{ij}(x_i - x_j)^2 = 0$. If G_n is connected, $\sum_{(i,j)\in E} a_{ij}(x_i - x_j)^2 = 0$ implies $x_1 = x_2 = ... = x_n$. Together with $\sum_{i=1}^n x_i = 0$, this implies $x_1 = x_2 = ... = x_n = 0$, which contradicts the fact that $x \neq 0$. Therefore the underlying graph G_n is not connected.

□

B. The Cauchy Interlace Theorem

Theorem 7 (Cauchy Interlace Theorem [17])**.** Let A be a Hermitian matrix of order n, and let B be a principal sub-matrix of A of order $n-1$. If $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n$ lists the eigenvalues of A and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-2} \leq \mu_{n-1}$ the eigenvalues of B, then $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n$.

C. The Courant-Fischer Theorem

The minimax and maximin characterization of eigenvalues of Hermitian matrices, known as Courant-Fischer theorem, is represented in the following.

Theorem 8 (Courant-Fischer Theorem)**.** Let M be a $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq ... \leq \lambda_k \leq \cdots \leq \lambda_n$ then

$$\begin{aligned}\lambda_k = \min_U \max_x \{ \frac{x^T M x}{x^T x} &: U \subset R^n, \dim(U) = k, \\ &x \in U = \text{span}(U), x \neq 0\}\end{aligned}$$

and

$$\begin{aligned}\lambda_k = \max_U \min_x \{ \frac{x^T M x}{x^T x} &: U \subset R^n, \dim(U) = n-k+1, \\ &x \in U = \text{span}(U), x \neq 0\}.\end{aligned}$$

D. Proof of Lemma 1

The proof follows the proof idea of [18] and extends the result to weighted graphs.

Proof. Denote the Laplacian of G_n by L_n and denote the Laplacian of the graph $G_{n-1}^{(v)}$ by $L_{n-1}^{(v)}$. Removing the row and column of L_n that correspond to node v, we have the principle sub-matrix denoted by $P_{n-1}^{(v)}$. Label the nodes of G_n by $\{1, 2, ..., n\}$. Suppose the label of node v is i. We note that $P_{n-1}^{(v)}$ and $L_{n-1}^{(v)}$ are different only in the diagonal terms as follows:

$$L_{n-1}^{(v)} = P_{n-1}^{(v)} - \text{diag}(\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n), \tag{12}$$

where α_k is the connection weight between the i^{th} (i.e. node v) and k^{th} node. Then

$$\alpha_{\max}^v = \max\{\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n\}.$$

For simplicity, let $\Lambda_v = \text{diag}(\alpha_1, ..., \alpha_{i-1}, \alpha_{i+1}, ..., \alpha_n)$. Then

$$L_{n-1}^{(v)} = P_{n-1}^{(v)} - \Lambda_v. \tag{13}$$

Fix $k \in \{1, 2, ..., n-1\}$. We denote $\mathbb{U}_k$ as the set of subspaces $\{U \subset R^n \text{ with } \dim(U) = n-k+1\}$. It follows from Courant-Fisher Theorem that,

$$\lambda_k(P_{n-1}^{(v)}) = \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T P_{n-1}^{(v)} x}{x^T x}; x \neq 0\},$$

Substituting $P_{n-1}^{(v)}$ by $L_{n-1}^{(v)} + \Lambda_v$ we have

$$\lambda_k(P_{n-1}^{(v)}) = \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T (L_{n-1}^{(v)} + \Lambda_v) x}{x^T x}; x \neq 0\}.$$

Further simplifying the right hand side, we obtain

$$\begin{aligned}\lambda_k(P_{n-1}^{(v)}) &\leq \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T (L_{n-1}^{(v)} + \alpha_{\max}^v I) x}{x^T x}; x \neq 0\} \\ &= \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T L_{n-1}^{(v)} x}{x^T x} + \alpha_{\max}^v; x \neq 0\} \\ &= \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T L_{n-1}^{(v)} x}{x^T x}; x \neq 0\} + \alpha_{\max}^v \\ &= \lambda_k(L_{n-1}^{(v)}) + \alpha_{\max}^v.\end{aligned}$$

By Cauchy Interlace Theorem, we have

$$\lambda_k(L_n) \leq \lambda_k(P_{n-1}^{(v)}).$$

Therefore we obtain the following inequality

$$\lambda_k(L_{n-1}^{(v)}) \geq \lambda_k(L_n) - \alpha_{\max}^v.$$

Hence

$$\lambda_k(G_{n-1}^{(v)}) \geq \lambda_k(G_n) - \alpha_{\max}^v, \quad 1 \leq k \leq n-1.$$

□

E. Proof of Lemma 2

Proof. Following the same set up in the proof of Lemma 1, we have

$$\lambda_k(P_{n-1}^{(v)}) = \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T (L_{n-1}^{(v)} + \Lambda_v) x}{x^T x}; x \neq 0\}.$$

Further simplifying the right hand side, we obtain

$$\begin{aligned}\lambda_k(P_{n-1}^{(v)}) &\geq \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T (L_{n-1}^{(v)} + \alpha_{\min}^v I) x}{x^T x}; x \neq 0\} \\ &= \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T L_{n-1}^{(v)} x}{x^T x} + \alpha_{\min}^v; x \neq 0\} \\ &= \max_{U\in\mathbb{U}_k} \min_{x\in U} \{ \frac{x^T L_{n-1}^{(v)} x}{x^T x}; x \neq 0\} + \alpha_{\min}^v \\ &= \lambda_k(L_{n-1}^{(v)}) + \alpha_{\min}^v.\end{aligned}$$

By Cauchy Interlace Theorem, we have

$$\lambda_{k+1}(L_n) \geq \lambda_k(P_{n-1}^{(v)}).$$

Therefore we obtain the following inequality

$$\lambda_{k+1}(L_n) \geq \lambda_k(L_{n-1}^{(v)}) + \alpha_{\min}^v.$$

Hence

$$\lambda_k(G_{n-1}^{(v)}) \leq \lambda_{k+1}(G_n) - \alpha_{\min}^v, \quad 1 \leq k \leq n-1.$$

$\square$

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