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Matrix convexity and noncommutative Nevanlinna–Pick interpolation

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Recent years saw a remarkable emergence of several general theories that are matrix or free noncommutative analogues of well established classical theories in the commutative setting. Since matrices are natural variables in many if not most problems in system theory, it is not surprising that these theories are often related to systems and control.

In the noncommutative (nc) setting we replace the say complex euclidean space $\mathbb{C}^d$ by the *nc space*, i.e., the disjoint union of square matrices of all sizes, over it:

$$(\mathbb{C}^d)_{\rm nc} = \coprod_{n=1}^{\infty} \left(\mathbb{C}^{n\times n}\right)^d.$$

The basic operations are direct sums: if $X = (X_1,\ldots,X_d) \in \left(\mathbb{C}^{n\times n}\right)^d$, $Y = (Y_1,\ldots,Y_d) \in \left(\mathbb{C}^{m\times m}\right)^d$,

$$X \oplus Y = (X_1 \oplus Y_1, \ldots, X_d \oplus Y_d) = \left(\begin{bmatrix} X_1 & 0 \\ 0 & Y_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_d & 0 \\ 0 & Y_d \end{bmatrix}\right) \in \left(\mathbb{C}^{n+m\times n+m}\right)^d,$$

and left and right multiplication by matrices: if $X = (X_1,\ldots,X_d) \in \left(\mathbb{C}^{n\times n}\right)^d$, $T \in \mathbb{C}^{m\times n}$, $S \in \mathbb{C}^{n\times m}$,

$$TXS = (TX_1S, \ldots, TX_dS).$$

A set $\Omega \subseteq (\mathbb{C}^d)_{\rm nc}$, $\Omega_n := \Omega \cap \left(\mathbb{C}^{n\times n}\right)^d$, is called matrix convex if for all $X^1,\ldots,X^k \in \Omega$, $X_i \in \Omega_{n_i}$, and all $V_i \in \mathbb{C}^{n_i\times n}$ with $\sum_{i=1}^k V_i^* V_i = I_n$, we have that

$$\sum_{i=1}^k V_i^* X_i V_i \in \Omega_n.$$

Matrix convex sets were introduced in the remarkable paper [4] that established for them the nc analogues of separation theorems with linear inequalities replaced by linear matrix inequalities. In particular, one can combine nc convexity with nc polynomial and rational inequalities, leading to the subject of nc convex algebraic geometry [8], [6], [7], motivated in particular by the applications of semidefinite programming to dimension independent problems in systems and control [5], [9]. The nc setting is often more rigid than the familiar commutative setting, allowing one to deduce surprisingly strong result. E.g., every nc basic closed semialgebraic set which is matrix convex is a nc spectrahedron, i.e., is given by a linear matrix inequality.

There is also a nc function theory that goes back to the pioneering work of Taylor on noncommutative spectral theory [11], [12], was further developed in [13], [14], and given a systematic exposition in [10]. A set $\Omega \subseteq (\mathbb{C}^d)_{\rm nc}$ is called a nc set if it is closed under direct sums: $X, Y \in \Omega$ implies $X \oplus Y \in \Omega$. A function f from a nc set $\Omega \subseteq (\mathbb{C}^d)_{\rm nc}$ to $\mathbb{C}_{\rm nc}$ with $f(\Omega_n) \subseteq \mathbb{C}^{n\times n}$ is called a nc function if

1) f respect direct sums: for all $X, Y \in \Omega$, $f(X \oplus Y) = f(X) \oplus f(Y)$;
2) f respects similarities: for all $n \in \mathbb{N}$, $X \in \Omega_n$ and $S \in \mathrm{GL}_n(\mathbb{C})$ such that $SXS^{-1} \in \Omega_n$, $f(SXS^{-1}) = Sf(X)S^{-1}$.

In this talk I will discuss the following result, which provides a solution to the Nevanlinna–Pick interpolation problem for nc functions on a matrix convex set:

Let $\Omega \subseteq (\mathbb{C}^d)_{\rm nc}$ be an open matrix convex set, let $\Lambda = (\Lambda_1,\ldots,\Lambda_d) \in \Omega_s$ and let $F \in \mathbb{C}^{s\times s}$. There exists a nc function $f\colon \Omega \to \mathbb{C}_{\rm nc}$ such that $f(\Lambda) = F$ and $\|f(X)\| \le 1$ for all $X \in \Omega$ if and only if:

1) F belongs to the algebra generated by $\Lambda_1,\ldots,\Lambda_d$;
2) for all $m \in \mathbb{N}$ and all $S \in \mathrm{GL}_{sm}(\mathbb{C})$ such that

$$S(\Lambda \otimes I_m)S^{-1} \in \Omega,$$

we have $\|S(F \otimes I_m)S^{-1}\| \le 1$.

This result follows from a much more general interpolation theorem of [3] (which generalizes previous results of [2] and [1]) that applies to any set of the form $\{X \in \Xi\colon \|Q(X)\| < 1\}$ where Q is a nc function on a bifull nc set $\Xi \subseteq \mathcal{V}_{\rm nc}$ (for a possibly infinite dimensional complex vector space $\mathcal{V}$) with values in $\mathcal{L}(\mathcal{R}, \mathcal{S})$ for some Hilbert spaces $\mathcal{R}$ and $\mathcal{S}$. In our case Q is the Cayley transform of a (in general infinite dimensional) linear matrix inequality defining a matrix convex set.

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