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Majorization Theory in Sensor Scheduling

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Extended Abstract

This brief paper presents that the majorization theory plays an essential role in a class of sensor scheduling problems, whose solutions all have periodic or uniformly distributed patterns. This brief paper revisits the problem of communication time scheduling for a single sensor with local computation capability, and strengthens its original result by the majorization theory.

The sensor scheduling problem to be studied exists in the networked control systems (NCSs), which has developed fast since the beginning of the 21 century. It intends to address the issue of limited communication resource existing in the practical applications, which may result from the unrechargeable battery supply for network component individuals, or limited bandwidth of wireless communication channels. As the resource is limited, the communications of sensors need to get scheduled so that some required performance of the system can be achieved when some criterion is specified. In this paper, state estimation quality is the performance of concern.

Among the studies in sensor scheduling, the solutions of a class of problems share a great similarity, where they all have periodic or uniformly distributed patterns. Hovareshti et al. [1] studies the scheduling of two sensors with local computing capability (the so-called smart sensor) and presented that the optimal schedule is use the two sensors alternatively and periodically. Shi et al. did a series of research on this field. They studied the scheduling problem of single smart sensor [2], of two normal sensors [3], and of one sensor observing two state processes [4]. The solution to the above problems are periodic. Yang and Shi [5] considered the scheduling of a single normal sensor in a first-order system within a finite time horizon, and proved that the necessary condition of a schedule to be optimal is that the transmission times are distributed as uniformly as possible. The uniformly distributed pattern also exists in the solution to the optimal communication channel allocation. Yang et al. [6] investigated the channel allocation among multiple identical sensors with local estimators and showed that the optimal allocation is that the channels connected to each sensor should be uniformly distributed.

The fact that the optimal solution has a periodic or uniform pattern implies that an expression as follows holds:

$$J(\bar{\omega}, \bar{\omega}, \ldots, \bar{\omega}) \leq J(\omega_1, \omega_2, \ldots, \omega_n),$$

where $\bar{\omega} = (1/n) \sum \omega_i$. This phenomenon is also observed and studied in various fields, such as the study of income inequality in economics [7], [8], liquid mixing in physics and chemistry [9], the representation theory of the symmetric groups in group theory [10], and so on. The intuition among these studies is the need to compare the degree of spreading or variation of the components of a vector.

Majorization, one advanced branch of inequality theory in mathematics, provides tools for this type of demands [11]. Important achievements were obtained by many researchers in this field. Hardy, Littlewood and Polya [12] first studies the comparison of a category of objective functions for two vectors, and showed that the objective value of one vector is smaller than that of the other if it is majorized by the other. Another important contribution is made by Schur [13].

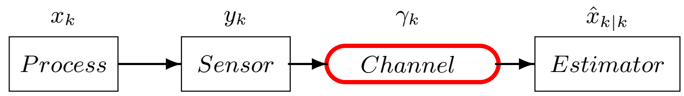

Fig. 1. A sensor measures a process and sends its data to a remote estimator via a controlled channel.

This paper intends to apply the majorization theory to the sensor scheduling problem. The benefit of the introduction of this theory is that the comparison of the performances of two schedules is replaced by the majorization order between the parameters of two schedules. Usually, the state estimation criterion is updated by the algebraic Riccati equation, which is difficult to deal with. When the estimation quality is not required to be extremely accurate, proper relaxation can be made. The majorization order enables a rough indication to the estimation performance and the Riccati equations are avoided. Moreover, since the intrinsic natures of the equations in state estimation (such as the convexity of algebraic Riccati function) is closely associated to the conditions of the majorization theory, the theory is likely to be applied to other problems in the field as a new powerful tool.

The contribution of the paper is as follows.

1) This paper applies the majorization theory to address the sensor scheduling problem and points out that majorization theory is the essence of a class of related problems whose solutions have periodic or uniform pattern.
2) This paper revisits a problem of scheduling a smart sensor and provides a stronger result. It enables the comparison of two non-optimal schedules and reduces it to compare their orders (if exist) in the sense of majorization.
3) The scheduling of a simple normal sensor in a general-order system is considered. It is relaxed to minimize the upper bound of objective function and optimal schedules is given.

A. Problem Setup

In this subsection we propose a communication time scheduling problem for a single sensor. The system to be considered is shown in Fig. 1. In the system, a dynamic process is measured by one sensor:

$$x_{k+1} = Ax_k + w_k, \tag{1}$$

$$y_k = Cx_k + v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the system state at time k and $y_k \in \mathbb{R}^m$ is the measurement taken by the sensor. $\{w_k\}$ and $\{v_k\}$ are zero-mean white Gaussian noise processes with $\boldsymbol{E}[w_k w_j'] = \delta_{kj} Q$ ($Q \geq 0$) and $\boldsymbol{E}[v_k (v_j)'] = \delta_{kj} R$ ($R > 0$). $\{w_k\}$ and $\{v_k\}$ are also independent processes, i.e., $\boldsymbol{E}[w_k (v_j)'] = 0,\ \forall j, k$. The initial state x_0 is a zero-mean Gaussian random vector uncorrelated to $\{w_k\}$ and $\{v_k\}$ for any k and has covariance $\Pi \geq 0$. We assume that the pair (C, A) is detectable.

At time k, the sensor transmits its measurement y_k to a remote estimator via a wireless channel. Due to limited communication resource, the sensor is unable to do the transmission at each time instant and needs to schedule the communication time. Define the *scheduling variable* γ_k as:

$$\gamma_k = \begin{cases} 1, & \text{to send } y_k, \\ 0, & \text{not to send } y_k. \end{cases}$$

Define a sensor schedule as $\theta \triangleq \{\gamma_k\}_1^T$ within the time horizon T.

The estimator calculates the minimum mean-squared error (MMSE) estimate of x_k based on the received sensor measurements up to time k. Denote the set of all the measurements received up to time k as $\tilde{\boldsymbol{Y}}_k = \{\gamma_1 y_1, \gamma_2 y_2, \ldots, \gamma_k y_k\}$. Define the *a priori* estimate and associated error covariance as

$$\begin{aligned} \hat{x}_{k|k-1} &\triangleq \boldsymbol{E}[x_k|\tilde{\boldsymbol{Y}}_{k-1}], \\ P_{k|k-1} &\triangleq \boldsymbol{E}[(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})'|\tilde{\boldsymbol{Y}}_{k-1}], \end{aligned}$$

and the *a posteriori* estimate and its associated error covariance as

$$\begin{aligned} \hat{x}_{k|k} &\triangleq \boldsymbol{E}[x_k|\tilde{\boldsymbol{Y}}_k], \\ P_{k|k} &\triangleq \boldsymbol{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})'|\tilde{\boldsymbol{Y}}_k]. \end{aligned}$$

The MMSE estimate of x_k to be calculated is equal to the conditional mean $\hat{x}_{k|k}$. The procedure of computing these quantities is given as follows. At time k, if y_k is received by the estimator, they are computed by a Kalman filter [14]:

$$\begin{aligned} \hat{x}_{k|k-1} &= A\hat{x}_{k-1|k-1}, && (3) \\ P_{k|k-1} &= AP_{k-1|k-1}A' + Q, && (4) \\ K_k &= P_{k|k-1}C'(CP_{k|k-1}C' + R)^{-1}, && (5) \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}), && (6) \\ P_{k|k} &= (I - K_k C)P_{k|k-1}, && (7) \end{aligned}$$

where the recursion starts from $\hat{x}_{0|0} = 0$ and $P_{0|0} = \Pi$. When the sensor does not send y_k, the update is simply

$$\begin{aligned} \hat{x}_{k|k} &= A\hat{x}_{k-1|k-1}, \\ P_{k|k} &= AP_{k-1|k-1}A' + Q. \end{aligned}$$

The estimation performance under limited measurement transmission is of concern. We investigate the average performance within the time horizon and consider the following objective function:

$$J(\theta) = \sum_{k=0}^{T} \mathrm{Tr}(P_{k|k}). \tag{8}$$

We consider the following problem:

Problem 1:

$$\begin{aligned} \min_{\theta} \quad & J(\theta) \\ \text{s.t.} \quad & \sum_{k=1}^{N} \gamma_k \leq d, \end{aligned}$$

where d ($d \in \mathbb{N}, d < T$) is the number of available transmission times of the sensor.

B. Majorization Theory in Sensor Scheduling

The results to be used given be the majorization theory are presented as follows.

Theorem 1 ([12]): $x \prec y$ if and only if for all convex functions $\psi : \mathbb{R} \to \mathbb{R}$,

$$\sum_{i=1}^{n} \psi(x_i) \leq \sum_{i=1}^{n} \psi(y_i).$$

Based on this theory, the following deduction is achieved.

Corollary 1: For a function $\phi : \mathbb{R}^n \to \mathbb{R}$ with

$$\phi(x) = \sum_{i=1}^{n} \psi(x_i)$$

where $\psi(\cdot)$ is convex, its minimum is given as

$$\phi^\star = \sum_{i=1}^{n} \psi(\bar{x}),$$

where $\bar{x} = (1/n)\sum_{i=1}^{n} \psi(x_i)$.

In next two subsections we apply the majorization theory in two models. First we revisit the problem of communication time scheduling for a smart sensor and strengthens its original result by the majorization theory. Secondly, we study the scheduling for a single normal sensor in a general-order system, and the optimal schedules for minimizing the upper bound of objective function is provided.

C. Scheduling of a Smart Sensor

In this subsection we revisit the problem studied in [2] to reveal its essential nature by majorization theory.

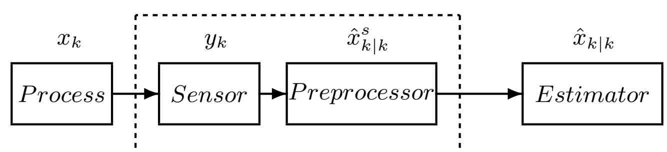

Fig. 2. Sensor scheduling of a smart sensor.

In the model (Fig. 2), the sensor is equipped a local computation part in addition to the model proposed in Section II. It is able to locally compute the MMSE estimate $\hat{x}^s_k$ and associated estimation error convariance P^s_k via the following set of equations:

$$\begin{aligned} \hat{x}^s_{k|k-1} &= A\hat{x}^s_{k-1|k-1}, \\ P^s_{k|k-1} &= AP^s_{k-1|k-1}A' + Q, \\ K_k &= P^s_{k|k-1}C'(CP^s_{k|k-1}C' + R)^{-1}, \\ \hat{x}^s_{k|k} &= A\hat{x}^s_{k-1} + K_k(y_k - CA\hat{x}^s_{k-1}), \\ P^s_{k|k} &= (I - K_k C)P^s_{k|k-1}, \end{aligned}$$

where the recursion starts from $\hat{x}^s_{0|0} = 0$ and $P^s_{0|0} = \Pi$. The estimation error covariance $P^s_{k|k}$ converges to a steady-state value exponentially fast:

$$\lim_{k\to\infty} P^s_{k|k} = \overline{P}.$$

For simplicity we assume that $P^s_{k|k} = \overline{P}$ from $k = 1$. The sensor sends $\hat{x}^s_{k|k}$ instead of y_k when the transmission is allowed. Define an operator $h : \mathbb{S}^n_+ \to \mathbb{S}^n_+$:

$$h(X) \triangleq AXA' + Q. \tag{9}$$

The state estimate $\hat{x}_{k|k}$ and error covariance $P_{k|k}$ are updated as follows:

$$(\hat{x}_{k|k}, P_{k|k}) = \begin{cases} \left(A\hat{x}_{k-1|k-1}, h(P_{k-1|k-1})\right), & \text{if } \gamma_k = 0, \\ (\hat{x}^s_{k|k}, \overline{P}), & \text{if } \gamma_k = 1. \end{cases}$$

Denote the set of transmission times as $\boldsymbol{\kappa} = \{k_1, k_2, \ldots, k_d\}$, where $\gamma_{k_i} = 1$, and the transmission intervals as $\omega_1 = k_1$, $\omega_i = k_i - k_{i-1},\ i = 2, \ldots, d$, and $\omega_{d+1} = T - k_d + 1$. The objective (8) becomes

$$J(\theta) = \sum_{j=0}^{d+1} \sum_{i=0}^{\omega_j} \mathrm{Tr}\left(h^i(\overline{P})\right).$$

The authors of [2] proves by mathematical calculation that the optimal schedule satisfies that the intervals ω_i should be almost uniformly distributed, i.e., the difference of the lengths is not larger than 1. We point out that it is the result of the majorization theory.

Define

$$\psi(x) = \sum_{i=0}^{l} \mathrm{Tr}\left(h^i(\overline{P})\right) + (x-l)h^{l+1}(\overline{P}), \quad x > 0, \tag{10}$$

where $l = \lfloor x \rfloor$, the largest number less than x. Notice that when x is a positive integer, $\psi(x) = \sum_{i=0}^{x} \mathrm{Tr}\left(h^i(\overline{P})\right)$.

Lemma 1 ([2]): For $0 \le t_1 \le t_2$, the following inequality holds:

$$h^{t_1}(\overline{P}) \le h^{t_2}(\overline{P}).$$

In addition, if $t_1 < t_2$, then

$$\mathrm{Tr}\left(h^{t_1}(\overline{P})\right) < \mathrm{Tr}\left(h^{t_2}(\overline{P})\right).$$

By Lemma 1 it is simple to verify that $\psi(x)$ is convex. Then

$$J(\theta) = \sum_{j=0}^{d+1} \psi(\omega_i),$$

which satisfies the conditions of Theorem 1. We have the following result:

Theorem 2: For two schedules θ and θ' with $\omega = (\omega_1, \omega_2, \ldots, \omega_{d+1})$ and $\omega' = (\omega'_1, \omega'_2, \ldots, \omega'_{d+1})$ respectively, $\omega \prec \omega'$ if and only if

$$J(\theta) \le J(\theta').$$

The theorem implies the following corollary.

Corollary 2: A schedule θ is optimal if and only if

$$\omega_i = m \text{ or } m+1, \ \ i = 1, 2, 3, \ldots, d+1,$$

where $m = \lfloor \frac{T+1}{d+1} \rfloor$.

This corollary is exactly the same result of [2], proved from the perspective of majorization.

D. Scheduling of a Simple Sensor with General Order

In this subsection we consider Problem 1. Since the state process is secured stable by the controller, we assume that A is stable. According to Lyapunov theory, there exists a unique $\tilde{P}$ satisfying

$$h(\tilde{P}) \quad = \quad A\tilde{P}A' + Q.$$

Define operators $\tilde{g}, g : \mathbb{S}^n_+ \to \mathbb{S}^n_+$:

$$\begin{aligned} \tilde{g}(X) &\triangleq X - XC'(CXC' + R)^{-1}CX, \\ g(X) &\triangleq \tilde{g}h(X). \end{aligned}$$

Then the update for the estimation error covariance is

$$P_{k|k} = \begin{cases} h(P_{k-1|k-1}), & \text{if } \gamma_k = 0, \\ g(P_{k-1|k-1}), & \text{if } \gamma_k = 1. \end{cases}$$

The objective is simplified as

$$J(\theta) = \sum_{j=0}^{d+1} \sum_{i=0}^{\omega_j} \mathrm{Tr}\left(h^i(P_{k_j})\right).$$

The problem of minimizing $J(\theta)$ is generally difficult, due to the complexity in high order Riccati equations. In this paper we only investigate the case where C is invertible, and leave the general case as a future work. Define

$$M \triangleq (C'R^{-1}C)^{-1}.$$

We study the case that $M \le Q$. Notice that M and Q represent the randomness degree of the measurements noise and the process noise respectively. The condition means the requirement that the measurement contains more precision. Define

$$\bar{J}(\theta) \triangleq \sum_{j=0}^{d+1} \sum_{i=0}^{\omega_j} \mathrm{Tr}\left(h^i(M)\right).$$

Lemma 2: $\bar{J}(\theta) \ge J(\theta)$.

We consider to find the schedules which minimizes the upper bound $\bar{J}(\theta)$.

Lemma 3: For $0 \le i \le j$, the following inequality holds:

$$h^i(M) \le h^j(M).$$

In addition, if $i < j$, then

$$\mathrm{Tr}\left(h^i(M)\right) < \mathrm{Tr}\left(h^j(M)\right).$$

Define

$$\xi(x) = \sum_{i=0}^{l} \mathrm{Tr}\left(h^i(M)\right) + (x-l)h^{l+1}(M), \quad x > 0,$$

where $l = \lfloor x \rfloor$. Lemma 2 secures the convexity of $\xi(x)$. Since $\bar{J}(\theta) = \sum_{j=0}^{d+1} \xi(\omega)$ satisfies the condition of Theorem 1, the following results holds.

Theorem 3: For two schedules θ and θ' with $\omega = (\omega_1, \omega_2, \ldots, \omega_{d+1})$ and $\omega' = (\omega'_1, \omega'_2, \ldots, \omega'_{d+1})$ respectively, $\omega \prec \omega'$ if and only if

$$\bar{J}(\theta) \le \bar{J}(\theta').$$

The schedule θ minimizes $\bar{J}(\theta)$ if and only if

$$\omega_i = m \text{ or } m+1, \ \ i = 1, 2, 3, \ldots, d+1,$$

where $m = \lfloor \frac{T+1}{d+1} \rfloor$.

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