23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

# Kalman Filtering with Intermittent Observations: A Structural Approach

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***Abstract*— In this extended abstract, we revisit the problem of Kalman filtering with intermittent observations. It is known that the definition of non-degeneracy plays a key role in determining the critical value of packet arrival probability for bounded estimate errors. In this extended abstract, we provide topological conditions for the existence of eigen-cycle, which can lead to degenerate systems. We prove that aperiodicity of the graph corresponding to the $A$ matrix is a necessary condition for non-existence of eigen-cycle. We also prove that a special class of aperiodic graphs does not contain non-zero eigen-cycles.**

***Index Terms*— Estimation, Kalman filtering, networked control**

***AMS Subject Classification*—93B07, 93B60**

## I. INTRODUCTION

Wireless sensor networks (WSNs) have attracted wide attention in recent years because of its successful applications in various domains, such as area monitoring, building automation, and navigation of an autonomous vehicle [1]. Due to the sensitivity of wireless connections to changes in the environment, packet loss is a common phenomenon for data traveling in WSNs, and its influence should not be neglected. Spurred by this consideration, a large amount of work has been devoted to design and analyze control and estimation algorithms over unreliable networks .

In this extended abstract, we focus on the Kalman filtering over unreliable networks in which the packets arrive according to a certain probability and thus the observations are intermittent. This problem is originally formulated and studied in [2], in which the authors consider discrete-time linear Gaussian systems under the assumption of a non-zero loss probability associated with each packet. It has been proven that, although the Kalman filter is still the optimal estimator in this packet loss situation, its estimate error may be unbounded if the arrival probability of the packets is below a critical value. The authors provide the upper and lower bounds of the critical value for general systems. However, the bounds are only tight for several special cases.

The study in [2] inspires a substantial amount of work to characterize the critical value of packet arrival probability in more general cases. Mo and Sinopoli also introduce a new concept of non-degeneracy, that requires the system to remain observable even if it is sampled at a different frequency [3]. They further prove that the critical value for a non-degenerate system is indeed the lower bound. Rohr et al. and Sui et al. extend the study by including degenerate systems and by considering a more general packet loss model [4], [5]. It has been shown that for degenerate systems, the critical value may be higher than the lower bounds. In other words, more communication resources are required to stabilize the system.

Even though the non-degeneracy is a desirable property to ensure the stable state estimation with bad communication quality, the condition for non-degeneracy depends on the rank of matrices, which is non-trivial to validate, especially for large scale networked systems. Furthermore, it does not provide any insight on how to design the system to avoid degeneracy. In this extended abstract, we utilize a concept, called eigen-cycle, to provide topological conditions to characterize non-degenerate systems from the graph-theoretic perspective.

The extended abstract is organized as follows: Section II reviews some results from the literature and formulates the problem. In Section III, we present our main results. Section IV concludes the extended abstract.

## II. PROBLEM FORMULATION

### A. System Settings

Consider the following LTI system:

$$x(k+1) = Ax(k) + w(k) \quad (1)$$
$$y(k) = Cx(k) + v(k) \quad (2)$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $y(k) \in \mathbb{R}^m$ is the measurement vector; $w(k)(v(k))$ is the process (measurement) noise, which is assumed to be zero mean and i.i.d. Gaussian with covariance $Q(R)$. The initial state $x(0)$ is zero mean Gaussian with covariance $\Sigma$. Assume that $w(k)$, $v(k)$, $x(0)$ are jointly independent, $Q$, $R$, $\Sigma$ are strictly positive, and $(A, C)$ is observable.

Consider the case in which observations are sent to the estimator via an unreliable communication channel, where the packet arrival is modeled by an i.i.d. Bernoulli random process $\{\gamma(k)\}$. According to this model, the measurement $y(k)$ sent at time $k$ reaches the estimator if $\gamma(k) = 1$; it is lost otherwise. Define the packet arrival rate to be $p \triangleq P(\gamma(k) = 1)$. We further assume that $\gamma(k)$ is independent of $w(k)$, $v(k)$ and $x(0)$, i.e., the communication channel is independent of both process and measurement noises.

This work was supported by the Academic Research Fund Tier 1 project under Grant No. M4011504.040 from the Ministry of Education, Singapore.

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### B. Kalman Filtering with Intermittent Observations

The Kalman Filter equations for the above system were derived in [2] and take the following form:

$$\hat{x}(k) = \hat{x}(k|k-1) + \gamma(k)K(k)(y(k) - C\hat{x}(k|k-1))$$
$$P(k) = P(k|k-1) - \gamma(k)K(k)CP(k|k-1)$$

where

$$\hat{x}(k|k-1) = A\hat{x}(k-1)$$
$$P(k|k-1) = AP(k-1)A^T + Q$$
$$K(k) = P(k|k-1)C^T(CP(k|k-1)C^T + R)^{-1}$$
$$\hat{x}(0) = 0, P(0) = \Sigma$$

### C. Existence of Critical Value

The following theorem establishes the existence of a critical packet arrival probability for the stability of the Kalman filter:

*Theorem 1:* [2] If $A$ is unstable, then there exists a critical value $p_c \in [0, 1]$ such that

1) If $p < p_c$, then there exists some initial condition $\Sigma$, such that $\sup \mathbb{E}P(k) = \infty$;

2) If $p > p_c$, then for any initial condition $\Sigma$, there exists a matrix $M(\Sigma)$, such that $\sup \mathbb{E}P(k) \leq M(\Sigma)$.

In general, the computation of the critical value $p_c$ is difficult. A lower bound of $p_c$ is given in [2]:

$$p_c \geq 1 - \frac{1}{\rho(A)^2},$$

where $\rho(A)$ is the spectral radius of $A$.

### D. Non-degenerate System and Eigen-Cycle

Now we will give an equivalent definition of non-degeneracy from [4]:

*Definition 2:* A pair of matrices $(A, C)$ is called non-degenerate if $A$ is diagonalizable and for any strictly positive integer $h$, $(A^h, C)$ is **observable**.

Notice that $A$ matrix can be decomposed as $A = diag(A_u, A_s)$, where $A_u$ contains all the unstable and critically stable mode and $A_s$ is strictly stable. Since the strictly stable mode does not affect the stability of the estimator [3], we can weaken our definition of non-degeneracy.

*Definition 3:* A pair of matrices $(A, C)$ is called weakly non-degenerate if $A_u$ is diagonalizable and for any strictly positive integer $h$, $(A^h, C)$ is **detectable**.

The following theorem establishes the critical value of a weakly non-degenerate system:

*Theorem 4:* For a system with $(A, C)$ weakly non-degenerate, the critical value $p_c$ is given by

$$p_c = 1 - \frac{1}{\rho(A)^2}.$$

For general unstable LTI systems, the above lower bound of critical value is not always tight [6], [7]. In other words, a higher communication successful rate is needed to stabilize the Kalman filter. As a result, weak non-degeneracy is a desirable property to reduce the communication constraint on the system. However, when dealing with large scale networked systems, checking weak non-degeneracy may be difficult due to numerical issues or parameter uncertainties. To avoid such a problem, in the next subsection, we seek to provide topological conditions for weak non-degeneracy.

Before continuing on, we shall introduce the concept of eigen-cycle, which is closely related to non-degeneracy.

*Definition 5:* A set of eigenvalues $\{\lambda_1, \ldots, \lambda_j\}$ is called an eigen-cycle of $A$ if there exists a strictly positive integer $h$, such that $\lambda_1^h = \ldots = \lambda_j^h$. A matrix $A$ is (weakly) aperiodic if $A$ $(A_u)$ is diagonalizable and does not contain any eigen-cycle.

It can be proved, using Hautus lemma [8], that if $(A, C)$ is observable (detectable) and $A$ is (weakly) aperiodic, then $(A, C)$ is (weakly) non-degenerate.

## III. Main Result

In this section, we will consider non-degeneracy from a graph-theory perspective, by first introducing the concept of structured system and the graph of the system matrix $A$.

*Definition 6:* A system is called structured if each entry of its matrices $A$ and $C$, is either a fixed zero or a free parameter. A property of a structured system is called generic if it is true for almost every value of the free parameters.

The graph $G = \{V, E\}$ corresponding to matrix $A$ has $n$ vertices representing $n$ states of the system, i.e., $V = \{1, \ldots, n\}$. If $a_{ij} \neq 0$, i.e., state $j$ has influence on state $i$, there exists a directed edge from vertex $j$ to vertex $i$. In other words

$$E = \{(j, i) : i, j \in V, \, a_{ij} \neq 0\}.$$

We say $i$ is reachable from $j$ if there exists a path, i.e., sequential directed edges, from $j$ to $i$. A strongly connected component is a maximal subgraph in which every vertex is reachable from every other vertex. A single vertex with a self-loop is also considered as a strongly connected component.

A cycle is defined a path such that the first vertex on the path is the same as the last. The size of the cycle is the number of vertices visited by the cycle.

The period of a strongly connected component is the greatest common divisor of the lengths of all cycles (closed directed paths) inside the component. If the period is one, the component is called aperiodic; otherwise it is periodic.

Based on the concept of reachability, we can partition the vertex set $V = \{1, 2, \cdots, n\}$ as

$$V = V_1 \cup \cdots \cup V_r \cup \{i_1\} \cup \cdots \cup \{i_s\}$$

where each $V_i$ is a strongly connected component and $i_j$ is a vertex that does not belong to any strongly connected component.

### A. Sufficient Conditions for the Existence of Eigen-Cycle

In this subsection, we consider sufficient conditions for the existence of eigen-cycle. We will first consider a graph with one strongly connected component and then generalize it to general graphs.

*Theorem 7:* Suppose the matrix $A$ corresponding to a strongly connected graph with period $h$. If $\lambda$ is an eigenvalue of $A$, then $\lambda\alpha$ is also an eigenvalue, for any $\alpha^h = 1$.

*Proof:* If the graph corresponding to $A$ is periodic with period $h > 1$, from [9] we know that the system matrix can be written in its Frobenius form (by properly numbering the vertices in the graph) as follows:

$$A = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_{h-1,h} \\ A_{h1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Suppose that $A$ has an eigenpair $(v, \lambda)$ such that $Av = \lambda A$, which can be written as follows:

$$\begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_{h-1,h} \\ A_{h1} & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{h-1} \\ v_h \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{h-1} \\ v_h \end{pmatrix}$$

One can prove that

$$A \begin{pmatrix} \alpha^h v_1 \\ \alpha v_2 \\ \vdots \\ \alpha^{h-2} v_{h-1} \\ \alpha^{h-1} v_h \end{pmatrix} = \alpha\lambda \begin{pmatrix} v_1 \\ \alpha v_2 \\ \vdots \\ \alpha^{h-2} v_{h-1} \\ \alpha^{h-1} v_h \end{pmatrix}$$

If $\alpha^h = 1$, then $\alpha\lambda$ is an eigenvalue of $A$. ■

For the more general case where the graph is not necessarily strongly connected, the graph can be divided into some strongly connected components and some single vertices which do not belong to any components.

*Proposition 8:* The matrix $A$ contains an eigen-cycle if one of its strongly connected components is periodic.

*Proof:* This proposition can be proved by rearranging the index of the vertices, such that the system matrix $A$ can be written in the following block upper triangular form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}$$

Each block on the diagonal represents a strongly connected component or a single vertex. The eigenvalues of $A$ are the collection of eigenvalues of $A_{ii}$, $i = 1, \cdots, k$, unrelated to off-diagonal blocks.

If any strongly connected component $A_{ii}$ is of period $h_i > 1$, by the result from Theorem 7, the matrix $A$ will contain an eigen-cycle. ■

We conjecture that the above sufficient condition for the existence of eigen-cycle is also necessary. However, this has not been proved yet and we would like to investigate it in the future.

Using Hautus Lemma, we can easily prove the following corollary:

*Corollary 9:* If $C$ is a rank 1 matrix and $A$ contains a periodic strongly connected component, then $(A, C)$ is degenerate.

### B. Case Study of A Class of Aperiodic Graphs

Next we will consider a special class of strongly connected graphs that ensures the weak aperiodicity of the matrix $A$. Consider the graph shown in Fig 1, that is covered by 2 connected cycles. Suppose that two cycles have lengths $\tilde{m}$ and $\tilde{n}$, respectively. To ensure the graph is aperiodic, the greatest common divisor of $\tilde{m}$ and $\tilde{n}$, $\gcd(\tilde{m}, \tilde{n}) = 1$. Without loss of generality, we assume $\tilde{m} < \tilde{n}$. The products of edge weights of the cycles are $p$ and $q$, respectively, which are free parameters. Denote the number of common vertices of the two cycles by $r < \tilde{m}$.

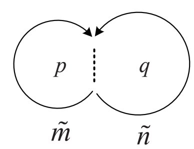


Fig. 1. An aperiodic graph with two joint cycles.

It can be proved that the characteristic polynomial of the system in Fig. 1 has the following form [10]:

$$\det(\lambda I - A) = \left(\lambda^{\tilde{n}} + p\lambda^{\tilde{n}-\tilde{m}} + q\right)\lambda^{\tilde{m}-r}$$

The roots of the characteristic polynomial are $\tilde{m} - r$ zeros and the roots of the following polynomial:

$$\begin{aligned} f(\lambda) &= \lambda^{\tilde{n}} + p\lambda^{\tilde{n}-\tilde{m}} + q \\ &\triangleq \lambda^n + p\lambda^m + q \end{aligned} \quad (3)$$

where redefining notations $n = \tilde{n}$ and $m = \tilde{n} - \tilde{m}$ is for the convenience of the following analysis.

*Theorem 10:* The $A$ matrix corresponding to the graph shown in Fig 1 is generically weakly aperiodic.

*Proof:* We prove the statement in two steps. First, we prove that $f$ almost surely does not contain double roots and therefore the unstable part of $A$ can be diagonalized almost surely. To this end, suppose $f$ has a double root $z$. Hence, $z$ is also a root of $f'$, i.e.,

$$nz^{n-1} + pmz^{m-1} = (nz^{n-m} + pm)z^{m-1} = 0.$$

Since $f(0) = q \neq 0$, we know that $nz^{n-m} + pm = 0$, which implies that $z = \alpha g(p)$, where $\alpha^{n-m} = 1$ and $g(p) = (pm/n)^{1/(n-m)}$.

Now use the fact that $f(z) = 0$, we have

$$q = -g(p)^n \alpha^n - pg(p)^m \alpha^m.$$

As a result, as long as $q$ does not take finitely many specific values, then $f(z) \neq 0$, which implies that $f$ almost surely does not have double roots.

Next we prove that for any $\alpha = \exp(j\theta)$ and $\alpha \neq 1$, the roots of $f(\lambda)$ and $f(\alpha\lambda)$ almost surely do no coincide. Suppose the opposite and

$$f(z) = z^n + pz^m + q = 0,\ f(\alpha z) = \alpha^n z^n + p\alpha^m z^m + q = 0.$$

Clearly

$$f(z) - f(\alpha z) = \left[(1-\alpha^n)z^{n-m} + p(1-\alpha^m)\right] z^m = 0,$$
$$\alpha^m f(z) - f(\alpha z) = (\alpha^m - \alpha^n)z^{n-m} + q(\alpha^m - 1) = 0.$$

Since $f(0) = q \neq 0$, we know that $z \neq 0$ and hence

$$(1-\alpha^n)z^{n-m} + p(1-\alpha^m) = 0,$$
$$(\alpha^m - \alpha^n)z^n + q(\alpha^m - 1) = 0.$$

which implies that

$$|1-\alpha^n| \times |z|^{n-m} = |p| \times |1-\alpha^m|,$$
$$|\alpha^m - \alpha^n| \times |z|^n = |q| \times |1-\alpha^m|. \quad (4)$$

Since $gcd(n,m) = gcd(\tilde{n},\tilde{m}) = 1$, we know that there exists integer $\tau$ and $\beta$, such that $\tau n + \beta m = 1$, which implies that

$$(\alpha^n)^\tau (\alpha^m)^\beta = \alpha \neq 1.$$

Therefore, $\alpha^n$ and $\alpha^m$ cannot be 1 simultaneously, which implies that (4) almost surely does not have a solution.

Let us define the set

$$\mathbb{A} = \{\alpha = \exp(j\theta) \neq 1 : \theta/\pi \text{ is rational.}\}$$

Consider the set of $(p,q)$, such that

$$S = \{(p,q) : (4) \text{ does not have a solution for any } \alpha \in \mathbb{A}\},$$

which can be written as the intersection of countably many sets:

$$S = \bigcap_{\alpha \in \mathbb{A}} \{(p,q) : (4) \text{ does not have a solution for } \alpha\}.$$

We can prove that the complement of $S$ has Lebesgue measure 0. Therefore, the unstable part of $A$ almost surely does not contain an eigen-cycle and hence $A$ is weakly aperiodic. ■

We can extend the result to more general graph, which can be covered by two connected cycles with coprime sizes.

*Corollary 11:* The $A$ matrix corresponding to a graph, which is covered by two connected cycles with coprime sizes, is generically weakly aperiodic.

From the relationship between weak non-degeneracy and weak aperiodicity, we know that the following holds.

*Corollary 12:* If $A$ matrix corresponds to a graph, which is covered by two connected cycles with coprime sizes, then the system is generically weakly non-degenerate. Moreover, the critical value of the system equals to the lower bound almost surely.

## IV. CONCLUSIONS

In this extended abstract we address the problem of Kalman filtering with intermittent observations. We provide structural conditions for aperiodicity of the $A$ matrix. We prove that aperiodicity of the graph generated by $A$ matrix is a necessary requirement of the aperiodicity of the $A$ matrix. We also prove that systems corresponding to a special kind of aperiodic graphs are generically weakly aperiodic and weakly non-degenerate.

For the future work, a complete characterization of the property of non-degeneracy from the graph-theoretical perspective is still an open problem. The analysis present herein can provide a valuable insight for achieving the final goal. In addition, designing graph-based algorithms to validate the structural conditions for non-degeneracy will also be investigated.

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