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Stability of constant gain output feedback for two-input systems with asynchronous sample and update timing*

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Abstract— **In this extended abstract, a stability certificate is presented for sampled-data implementations of static output feedback control with asynchronous sensor sampling and zero-order-hold actuator updates, given continuous-time two-input-single-output linear time-invariant system dynamics. The approach involves structured integral quadratic constraint (IQC) based robustness analysis, given individual bounds on the uncertain sample and two update intervals involved, and relationships between these. New IQCs for an operator that depends on multiple time-varying delays are also presented.**

I. Introduction

Consider the interconnection of a continuous-time system with two inputs, and one output, and a stabilizing static output feedback controller. A digital implementation of the feedback loop, as may be required when sensor output and actuator inputs cannot be otherwise interconnected, would generically involve sampling of the system output and update of the system inputs at discrete instants in times. This inherently gives rise to time-varying dynamics. Specifically, at the times between updates, each plant input is held constant at a value determined by the most recent plant output sample, taken at a varying time in the past [1].

Closed-loop stability is considered in the case where resource limitations result in asynchronous update and sample event sequences that satisfy individual bounds on the variable inter-update interval for each input, and inter-sample interval for the output, and relationships between these. The main result is an integral quadratic constraint (IQC) based stability certificate for the time-varying feedback interconnection given such bounds. The main contribution relates to the correspondingly structured characterization of asynchrony between the single plant output sample and two input update sequences. The input-output context of the IQC approach (see [2]) is also a distinguishing feature relative to much of the literature, which is based mainly on hybrid/impulsive state-space modelling and Lyapunov stability analysis [4]–[11]. In the state-space literature, it is standard to relate all sample and update events to a single time sequence, for which an interval bounds holds. By contrast, in the structured approach developed below, event sequences for the output and each input are characterized individually, and with respect to each other. Other examples of input-output analysis include [12] and [13], where sample/update timing is synchronous (albeit aperiodic), and earlier work [14] and [15], where direct feedthrough is excluded, and updates of the different components of the input is synchronous.

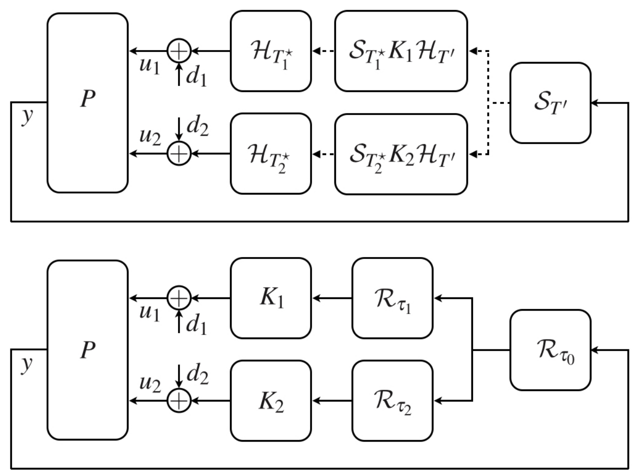

Fig. 1. Constant gain feedback for two-input plant with asynchronous update and sample timing (top); equivalent purely continuous-time system (bottom).

The feedback interconnection shown in the top part of Fig. 1 can be used to model the aforementioned asynchronous sampled-data implementation of a constant gain feedback path. This interconnection is to be interpreted as follows. At times in the set $T' = \{t'_k\}_{k=0}^{\infty}$, with $t'_0 = 0$, the system output y is sampled and the value of a buffer that stores the most recent sample is updated to the corresponding value.[1] This corresponds to the composite action of the operators denoted by $\mathcal{S}_{T'}$ and $\mathcal{H}_{T'}$. Respectively, these operators generate a sequence from samples of the continuous-time input at the set of times T', and a held constant between neighbouring instants of time in T' according to the corresponding input sample sequence. For $j \in \{1,2\}$, at each time in the set $T_j^\star = \{t_{j,k}^\star\}_{k=1}^{\infty}$, the j-th system input updates to a new constant value equal to the product of the static gain K_j and the contents of the sample buffer at the corresponding time.[2] This corresponds to the compositie action of the operator of multiplication by feedback gain K_j,

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[1]To simplify the development, suppose that there is no delay between the sample time and buffer update time, although this can be readily accommodated via appropriate adjustment of the analysis.

[2]Again, for simplicity it is assumed that there is no delay between time of plant input update and time of update computation, although it is possible to accommodate such delay within the analysis framework.

followed by $\mathcal{S}_{T_j^\star}$, and finally $\mathcal{H}_{T_j^\star}$, where the latter denote sample and hold operators with respect to the time sequence $T_j^\star$, in line with the description of $\mathcal{S}_{T'}$ and $\mathcal{H}_{T'}$ above.

Non-uniformity of the sample and update intervals, and asynchrony between sample and update events, are represented by bounds on the difference between relevant elements of the time sequences T', $T_1^\star$, and $T_2^\star$. Specifically, for the purpose of stability analysis, it is assumed that there exists constants $\tau', \tau_j^\star, \tau_j^\circ, \tau^\sharp \in (0,\infty)$ such that

$$\begin{aligned} &0 < t'_{k+1} - t'_k \le \tau', \quad 0 < t^\star_{j,k+1} - t^\star_{j,k} \le \tau_j^\star, \\ &\min T_j^\star \cap [t'_k, \infty) - t'_k \le \tau_j^\circ, \quad \text{and} \\ &\min T_{\bar{j}}^\star \cap [t^\star_{j,k}, \infty) - t^\star_{\bar{j},k} \le \tau^\sharp, \text{ for } k = 0,1,\ldots, \end{aligned} \tag{1}$$

with $t'_k \to \infty$, $t^\star_{j,k} \to \infty$, $\bar{j} = (j \bmod 2+1)$ and $j \in \{1,2\}$. The approach to certifying stability of the asynchronous sampled-data feedback interconnection is to devise a condition that implies stability for all realizations of T', $T_1^\star$, and $T_2^\star$, that comply with the structured collection of bounds in (1). This robust stability problem can be tackled by using IQCs to characterize the possibilities, and by applying the corresponding framework for the study of uncertain feedback interconnections described in [2], as elaborated below. As part of this, new IQCs are presented for a structured operator that depends on the time-varying delay operators $\mathcal{R}_j$ in the equivalent interconnection in the bottom part of Fig. 1, $j \in \{0,1,2\}$. Within the given setup, these time-varying delays do not necessarily reset to zero, which is another distinguishing feature of this work; see Remark 1 below. It may be conservative to consider all possible realizations of T', $T_1^\star$, and $T_2^\star$. This is the price of tractable analysis conditions for the time-varying interconnection.

The rest of the paper is organized as follows. First some notation and terminology are established to facilitate formulation of the aforementioned robust stability problem. Then a loop transformation is used in Section III to arrive at a problem that is tractable within the IQC analysis framework. New IQCs for characterizing a corresponding structured operator that depends on two time-varying delays are devised in Section IV to enable application of the IQC robustness result. Some concluding remarks are made in Section V.

II. Preliminaries

A. Signals, systems, and IQCs

The non-negative integers and reals are denoted by $\mathbb{N}_0$ and $\mathbb{R}$, and $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. The space of square integrable functions defined on $[0,\infty) \subset \mathbb{R}$ is denoted by $\mathbf{L}_2$ along with the usual norm and inner product $\|\cdot\|_{\mathbf{L}_2}$ and $\langle\cdot,\cdot\rangle_{\mathbf{L}_2}$. The extended $\mathbf{L}_2$ signal space is denoted by $\mathbf{L}_{2\mathrm{e}}$. This comprises functions f that satisfy $\pi_\tau f \in \mathbf{L}_2$ for $\tau > 0$, where π_τ denotes the truncation operator; $(\pi_\tau f)(t) = f(t)$ for $t \le \tau$, and $(\pi_\tau f)(t) = 0$ otherwise. The n-times cartesian products of $\mathbf{L}_2$ and $\mathbf{L}_{2\mathrm{e}}$ are denoted by $\mathbf{L}_2^n$ and $\mathbf{L}_{2\mathrm{e}}^n$. Often the spatial dimension of a signal space is dropped for convenience.

The operator $G : \mathbf{L}_{2\mathrm{e}} \to \mathbf{L}_{2\mathrm{e}}$ is said to be causal if $\pi_\tau G \pi_\tau - G\pi_\tau = 0$ for all $\tau > 0$. The restriction to $\mathbf{L}_2$, also denoted by G, is bounded if $\|G\| = \sup_{u \in \mathbf{L}_2} \|Gu\|_{\mathbf{L}_2} / \|u\|_{\mathbf{L}_2}$ is finite. If G is casual and bounded, then it is called stable. When G is linear and bounded, the adjoint is denoted by G^*; i.e., $\langle v, Gu\rangle_{\mathbf{L}_2} = \langle G^* v, u\rangle_{\mathbf{L}_2}$. If $G = G^*$, then G is said to be self-adjoint, in which case the notation $G \ge 0$ means $\langle u, Gu\rangle_{\mathbf{L}_2} \ge 0$ for all $u \in \mathbf{L}_2$. If G is linear and commutes with the forward shift operator (i.e., G is "time invariant"), then it can be represented as multiplication in the frequency domain by a transfer function matrix, also denoted by G for convenience. When G is also bounded on $\mathbf{L}_2$, this function is analytic and bounded in the right-half plane [16]. In this case, $\|G\| = \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \sigma_{\max}(G(\mathrm{j}\omega))$, and $G \ge 0$ if and only if $G(\mathrm{j}\omega) \ge 0$ for all $\omega \in [0,\infty]$, where $\mathrm{j} = \sqrt{-1}$ and $\sigma_{\max}(\cdot)$ denotes maximum singular value. If G admits a rational transfer function $G(s) = C(sI - A)^{-1}B + D$, the (non-unique) matrices (A,B,C,D) are called a state-space realization of G.

Given the measurable and uniformly bounded multiplier $\Pi = ((\omega \in \bar{\mathbb{R}}) \mapsto (\Pi(\mathrm{j}\omega) \in \mathbb{C}^{(m+p)\times(m+p)}))$, with $\Pi(\mathrm{j}\omega) = \Pi(\mathrm{j}\omega)^*$, the stable (i.e., causal and bounded) operator $\Delta : \mathbf{L}_{2\mathrm{e}}^p \to \mathbf{L}_{2\mathrm{e}}^m$ is said to satisfy the IQC defined by Π if

$$\int_{-\infty}^{\infty} \left(\begin{bmatrix} \widehat{v} \\ \widehat{\Delta v} \end{bmatrix} (\mathrm{j}\omega) \right)^* \Pi(\mathrm{j}\omega) \begin{bmatrix} \widehat{v} \\ \widehat{\Delta v} \end{bmatrix} (\mathrm{j}\omega) \, \mathrm{d}\omega \ge 0$$

holds for all $v \in \mathbf{L}_2^p$, where $\widehat{\cdot}$ denotes the Fourier transform. The multiplier Π is often block partitioned according to the components of v and Δv. Dependence of the multiplier on a parameter X is denoted by $\Pi(X)$.

Finally, given causal $G : \mathbf{L}_{2\mathrm{e}}^m \to \mathbf{L}_{2\mathrm{e}}^p$ and $\Delta : \mathbf{L}_{2\mathrm{e}}^p \to \mathbf{L}_{2\mathrm{e}}^m$, if for every $(w_1, w_2) \in \mathbf{L}_{2\mathrm{e}}^m \times \mathbf{L}_{2\mathrm{e}}^p$ there exist unique $(v_1, v_2) \in \mathbf{L}_{2\mathrm{e}}^m \times \mathbf{L}_{2\mathrm{e}}^p$ so that

$$\begin{cases} v_1 = \Delta v_2 + w_1 \\ v_2 = G v_1 + w_2 \end{cases}, \tag{2}$$

and the closed-loop map $[G,\Delta] = ((w_1,w_2) \in \mathbf{L}_{2\mathrm{e}}^m \times \mathbf{L}_{2\mathrm{e}}^p) \mapsto ((v_1,v_2) \in \mathbf{L}_{2\mathrm{e}}^m \times \mathbf{L}_{2\mathrm{e}}^p))$ is causal, then the feedback interconnection is called well-posed. Moreover, if the induced norm of the restriction to $\mathbf{L}_2$ is also bounded (i.e., $\|[G,\Delta]\| < \infty$), then the closed-loop is said to be stable.

B. Sample, hold, and delay operators.

Given $T = \{t_k\}_{k=0}^\infty \subset [0,\infty)$, satisfying $t_0 = 0$, $t_{k+1} - t_k > 0$, and $\lim_{k\to\infty} t_k = \infty$, the sampling operator $\mathcal{S}_T$ denotes the map from the continuous-time signal $v \in \mathscr{C}_r \cap \mathbf{L}_{2\mathrm{e}}$ to the discretely indexed sequence $\tilde{v} = \{\tilde{v}_k\}_{k=0}^\infty$, such that $\tilde{v}_k = v(t_k)$, where $\mathscr{C}_r$ denotes the space of right continuous functions defined on $[0,\infty)$. Conversely, $\mathcal{H}_T$ denotes the zero-order hold operator synchronized to the event sequence T, which maps the discretely indexed signal $\tilde{v} = \{\tilde{v}_k\}_{k=0}^\infty$ to the continuous-time signal v such that

$$v(t) = (\mathcal{H}_T \tilde{v})(t) = \tilde{v}_k \qquad \text{for } t \in [t_k, t_{k+1}),\ k \in \mathbb{N}_0;$$

note that $v \in \mathbf{L}_{2\mathrm{e}}$ because every finite truncation of the sequence $\tilde{v}$ is square summable. Given the function $\tau : [0,\infty) \to [0,\infty)$, the time-varying delay operator $\mathcal{R}_\tau : \mathbf{L}_{2\mathrm{e}} \to \mathbf{L}_{2\mathrm{e}}$ is defined by

$$(\mathcal{R}_\tau v)(t) = \begin{cases} v(t - \tau(t)) & \text{if } t - \tau(t) \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1: Given $T = \{t_k\}_{k=0}^{\infty}$, such that $0 < t_{k+1} - t_k \leq h$ for all $k \in \mathbb{N}_0$ and $\lim_{k\to\infty} t_k = \infty$, let $n(t) = \max\{k \mid t_k \in [0,t],\ k \in \mathbb{N}_0\}$ and $\tau(t) = t - t_{n(t)}$ for $t \in [0,\infty)$. Then

$$\mathcal{R}_\tau u = \mathcal{H}_T \mathcal{S}_T u \quad \text{for all } u \in \mathscr{C}_r \cap \mathbf{L}_{2e}. \tag{3}$$

Lemma 2: Given $\tau : [0,\infty) \to [0,\infty)$, the time-varying delay operator $\mathcal{R}_\tau$ commutes with multiplication by the constant gain K; i.e., $K\mathcal{R}_\tau - \mathcal{R}_\tau K = 0$.

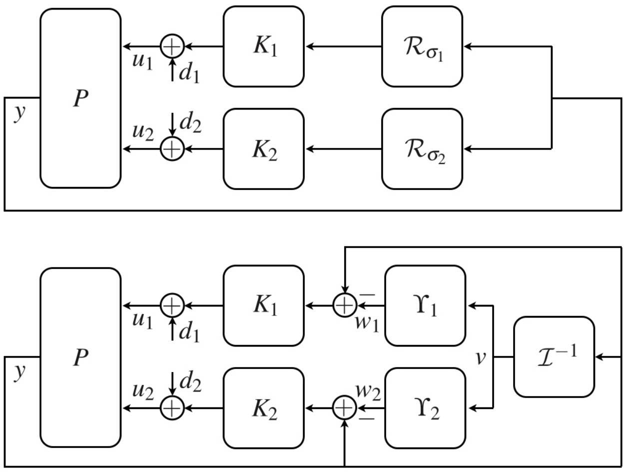

Fig. 2. Loop transformation of system shown in Fig. 1 to facilitate analysis in terms of IQC descriptions of the time-varying operators Υ_1 and Υ_2.

III. Closed-loop stability verification

Consider the feedback interconnection shown in the top part of Fig. 1. For $j \in \{1,2\}$, let $\tau', \tau_j^\star, \tau_j^\circ, \tau^\sharp \in (0,\infty)$ be bounds that characterize the possibly aperiodic and asynchronous time sequences $T' = \{t'_k\}_{k=0}^{\infty}$ and $T_j^\star = \{t_{j,k}^\star\}_{k=0}^{\infty}$ as shown in (1). Let the linear time-invariant system $P : \mathbf{L}_{2e} \times \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ have the strictly-proper rational transfer function

$$P(s) = \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix} = C_\mathrm{p}(sI - A_\mathrm{p})^{-1} \begin{bmatrix} B_{\mathrm{p}1} & B_{\mathrm{p}2} \end{bmatrix},$$

and the feedback gains K_1 and K_2 make the standard continuous-time closed-loop $[\begin{bmatrix} P_1 & P_2 \end{bmatrix}, \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}]$ stable.

Since all open-loop components are causal, and P has a strictly proper transfer function, the open loop is strongly causal, whereby the sampled-data feedback interconnection is well-posed in the following sense [17]: For every $d_1, d_2 \in \mathbf{L}_{2e}$, there exist unique $u_1, u_2 \in \mathbf{L}_{2e}$ that satisfy

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \left(I - \begin{bmatrix} \mathcal{H}_{T_1^\star}\mathcal{S}_{T_1^\star}K_1 \\ \mathcal{H}_{T_2^\star}\mathcal{S}_{T_2^\star}K_2 \end{bmatrix} \mathcal{H}_{T'}\mathcal{S}_{T'} \begin{bmatrix} P_1 & P_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and the closed-loop map $(d_1,d_2) \mapsto (u_1,u_2)$ is causal. The strict properness of P also implies the signal y is piecewise continuously differentiable. The closed-loop is stable if the restriction of the causal map $(d_1,d_2) \mapsto (u_1,u_2)$ to $\mathbf{L}_2$ is bounded. Note that the injection of a right-continuous signal $d_3 \in \mathscr{C}_r \cap \mathbf{L}_{2e}$ at the output of P can be modelled in terms of a corresponding $(d_1,d_2) \in \mathbf{L}_{2e}$ by linearity; n.b., $\mathcal{H}_{T_j^\star}\mathcal{S}_{T_j^\star}K_j\mathcal{H}_{T'}\mathcal{S}_{T'}d_3 \in \mathbf{L}_{2e}$ for $j \in \{1,2\}$. Moreover, if $d_3 \in \mathscr{C}_r \cap \mathbf{L}_2$, then $\mathcal{H}_{T_j^\star}\mathcal{S}_{T_j^\star}K_j\mathcal{H}_{T'}\mathcal{S}_{T'}d_3 \in \mathbf{L}_2$. As such, it suffices to consider the closed-loop map $(d_1,d_2) \mapsto (u_1,u_2)$.

The first result in this section pertains to the equivalence between all of the feedback interconnections shown in Fig. 1 and Fig. 2. In the latter, the operators $\mathcal{I} : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ and $\mathcal{I}^{-1} : \mathscr{D} \to \mathbf{L}_{2e}$ denote integration and differentiation. In particular, $\mathcal{I} = (v \in \mathbf{L}_{2e}) \mapsto ((t \mapsto \int_0^t v(x)\,\mathrm{d}x) \in \mathbf{L}_{2e})$, $\mathcal{I}\mathcal{I}^{-1}y = y$ for all y in the subset $\mathscr{D} \subset \mathbf{L}_{2e}$ of piecewise continuously differentiable functions, and $\mathcal{I}^{-1}\mathcal{I} = \mathrm{Id}$, where Id denotes the identity on $\mathbf{L}_{2e}$. Moreover, for $j \in \{1,2\}$ and $t \in [0,\infty)$, the following notation applies:

$$\begin{aligned}
&\tau_0(t) = t - t'_{n_0(t)}; \qquad \tau_j(t) = t - t^\star_{j,n_j(t)};\\
&n_0(t) = \max\{k \mid t'_k \in [0,t],\ k \in \mathbb{N}_0\};\\
&n_j(t) = \max\{k \mid t^\star_{j,k} \in [0,t],\ k \in \mathbb{N}_0\};\\
&\sigma_j(t) = \tau_j(t) + \tau_0(t - \tau_j(t)) = t - t'_{m_j(t)};\\
&m_j(t) = \max\{k \mid t'_k \in [0, t^\star_{j,n_j(t)}],\ k \in \mathbb{N}_0\};
\end{aligned} \tag{4}$$

and

$$\Upsilon_j = (\mathrm{Id} - \mathcal{R}_{\sigma_j})\mathcal{I}. \tag{5}$$

Remark 1: Note that σ_j is a saw-tooth function, with discontinuities that DO NOT necessarily reset to zero at the update times $T_j^\star$. This is a distinguishing feature of this work.

Lemma 3: The sampled-data feedback interconnection shown in the top part of Fig. 1 is equivalent to

$$\begin{cases} v = G_{vw}w + G_{vd}d \\ w = \Delta v \end{cases}, \tag{6}$$

with exogenous input d, and internal signals v and w, where

$$\begin{aligned}
\Delta &= \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix},\\
G_{vd} &= \mathcal{I}^{-1}(1 - (P_1K_1 + P_2K_2))^{-1} \begin{bmatrix} P_1 & P_2 \end{bmatrix}, \text{ and}\\
G_{vw} &= -\mathcal{I}^{-1}(1 - (P_1K_1 + P_2K_2))^{-1} \begin{bmatrix} P_1K_1 & 0 \\ 0 & P_2K_2 \end{bmatrix},
\end{aligned} \tag{7}$$

which is the last feedback interconnection shown in Fig. 2.

The feedback interconnection (6) is directly amenable to a standard robust stability analysis technique given an IQC characterization of the structured time-varying operator Δ.

Theorem 1: The operators G_{vw}, G_{vd}, and Δ, in (7) are stable (i.e., causal and bounded), and the feedback interconnection $[G_{vw}, \tau\Delta]$ is well-posed for $\tau \in [0,1]$. Moreover, given multiplier $\Pi = \Pi^*$, with $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$, if

- Δ satisfies the IQC defined by Π, and
- there exists $\varepsilon > 0$ such that

$$\left\langle \begin{bmatrix} G_{vw}w \\ w \end{bmatrix}, \Pi \begin{bmatrix} G_{vw}w \\ w \end{bmatrix} \right\rangle_{\mathbf{L}_2} \leq -\varepsilon \|w\|_{\mathbf{L}_2}^2 \ \forall\ w \in \mathbf{L}_2, \tag{8}$$

then $[G_{vw}, \tau\Delta]$ is stable for all $\tau \in [0,1]$. In particular, the feedback interconnection (6) is stable in this case.

Proof: By assumption $\begin{bmatrix} K_1^\top & K_2^\top \end{bmatrix}^\top$ stabilizes the strictly proper system P, whereby $(I - (P_1K_1 + P_2K_2))^{-1}\mathrm{diag}(P_1K_1, P_2K_2)$ is stable with strictly proper transfer function, whence $\mathcal{I}^{-1}(I - (P_1K_1 + P_2K_2))^{-1}\mathrm{diag}(P_1K_1, P_2K_2)$ is proper; i.e., G_{vw} is causal and bounded, as claimed. Similarly, G_{vd} is stable. Clearly, $\Upsilon_j = (I - \mathcal{R}_{\sigma_j})\mathcal{I} : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$ is causal, as it is the composition of causal operators. Boundedness of the restriction to $\mathbf{L}_2$ is established as part of Lemma 4 below. Moreover, for $j \in \{1,2\}$,

$$(\Upsilon_j v)(t) = \int_{t'_{m_j(t)}}^{t} v(x)\,\mathrm{d}x, \tag{9}$$

where $m_j(t)$ is defined in (4). As such, the operator Υ is strongly causal because for every $t > 0$, which implies the same for $\tau\Delta$ with $\tau \in [0,1]$. Therefore, $[G_{vw}, \tau\Delta]$ is well-posed [17]. With this established, the rest of the stated result follows by direct application of the standard IQC robust stability theorem; e.g., see [2]. ■

IV. IQCs for Δ

A family of IQCs for characterizing the uncertain operator Δ in (7) is developed below to enable application of the robust stability conditions in Theorem 1. First, IQCs are derived for the operators Υ_j individually, given corresponding bounds (1), which relate to constraints on the sample and update times $T' = \{t'_k\}_{k=0}^{\infty}$ and $T_j^\star = \{t^\star_{j,k}\}_{k=0}^{\infty}$. These are then combined to construct IQCs for Δ.

Lemma 4 (Bounded Gain): Given $j \in \{1,2\}$, if $0 < (t^\star_{j,k+1} - t^\star_{j,k}) \leq \tau_j^\star$ and $(\min T_j^\star \cap [t'_k, \infty) - t'_k) \leq \tau_j^\circ$ for $k \in \mathbb{N}_0$, then $\|\Upsilon_j\| \leq 2\tau_j^\star/\pi + \sqrt{\tau_j^\star \tau_j^\circ}$, where π is half the circumference of the unit circle.

Lemma 5 (Passivity): Given any $v \in \mathbf{L}_2$, the following holds with $w = \Upsilon_j v$: $\langle w, v\rangle_{\mathbf{L}_2} + (\tau_j^\circ/2)\|v\|^2_{\mathbf{L}_2} \geq 0$.

The relationship between Υ_1 and Υ_2 when acting on the same signal can also be used in the construction of an IQC for the structured Δ in (7).

Lemma 6: For all $v \in \mathbf{L}_2$, the following holds with $w_1 = \Upsilon_1 v$ and $w_2 = \Upsilon_2 v$: $\|w_2 - w_1\|^2_{\mathbf{L}_2} \leq (\tau' + \tau^\sharp)\tau^\sharp \|v\|^2_{\mathbf{L}_2}$.

Finally, based on Lemmas 4, 5 and 6, a family of IQCs can be derived for the operator Δ in (7).

Theorem 2: Given bounds $\tau', \tau_j^\star, \tau_j^\circ, \tau^\sharp \in (0,\infty)$ such that (1) holds for $j \in \{1,2\}$, let $\gamma_j^\star = (2\tau_j^\star/\pi + \sqrt{\tau_j^\star \tau_j^\circ})^2$. With Υ_j as in (5) for $j \in \{1,2\}$, the structured operator

$$\Delta = \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}$$

satisfies the IQC defined by the multiplier

$$\Pi(X_1, \cdots, X_5) = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^\top & \Pi_{22} \end{bmatrix}$$

for any choice of the parameters $X_i \geq 0$, $i = 1, \ldots, 5$, where

$$\Pi_{11} = \gamma_1^\star X_1 + \gamma_2^\star X_2 + \tau_1^\circ X_3 + \tau_2^\circ X_4 + (\tau' + \tau^\sharp)\tau^\sharp X_5,$$
$$\Pi_{12} = \begin{bmatrix} X_3 & X_4 \end{bmatrix}, \quad \text{and}$$
$$\Pi_{22} = \begin{bmatrix} -X_1 - X_5 & X_5 \\ X_5 & -X_2 - X_5 \end{bmatrix}.$$

Remark 2: Given $(A_{\mathrm{gw}}, B_{\mathrm{gw}}, C_{\mathrm{gw}}, D_{\mathrm{gw}})$, a state-space realization of the transfer function for G_{vw} in (7), the stability certificate (8) in Theorem 1 can be reformulated as a convex feasibility problem expressed in terms of finite-dimensional linear matrix inequalities by application of the KYP lemma (see [19]) to the equivalent frequency domain condition

$$\begin{bmatrix} G_{vw}(\mathrm{j}\omega) \\ I \end{bmatrix}^* \Pi(X_1, \ldots, X_5) \begin{bmatrix} G_{vw}(\mathrm{j}\omega) \\ I \end{bmatrix} < 0 \;\forall\; \omega.$$

V. Conclusion

A computationally tractable stability certificate is devised for an asynchronous feedback interconnect. The approach involves structured IQC based robust stability analysis given individual bounds on the inter-sample and inter-update intervals, and relationships between these. New IQCs are given for a related operator that depends on multiple time-varying delays. Numerical examples, performance analysis, and the impact of delay and quantization in the information exchange and processing, are all topics of ongoing work.

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