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A Convex Approach to the Finite Dimensional Matching Problem in Communication Systems*

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***Abstract*— In this work, the impedance matching problem in communication systems is formulated as a convex optimization problem. Important properties of the problem including feasibility, existence and uniqueness of solution are discussed. A practical example of matching filter synthesis using this technique is presented.**

I. INTRODUCTION

The problem of impedance matching in communication systems is to minimize the power reflection that is to be transmitted, by a generator, to a given load within a frequency band. The matching and filtering requirements in classical communication systems are usually satisfied by using a matching circuit followed by a bandpass filter. We propose here to design finite degree matching filters that integrates both, matching and filtering requirements, in a single circuit and thereby increase the overall efficiency and compactness of the system.

The foundational work in matching theory goes back to the fifties where Fano and Youla developed a synthesis procedure for matching networks [1]. It is based on the use of Darlington's two port equivalent networks, yielding a number of necessary conditions to be satisfied by the functions representing the reflection coefficient of the total network. These interpolations conditions however compromise the convexity of the associated optimization problem and led to practical applications only for loads of limited complexity and for reflection coefficients belonging to restricted classes: Tchebychev, Butterworth etc...This method was therefore progressively replaced by a non-convex optimization method called real frequency technique introduced by Carlin [2]. Even though this method yields reasonable results in practice, no results are known about the global optimality of the obtained matching network. More recently, in the eighties, J.W Helton proposed a more general approach using non-euclidean functional analysis [3]. In the latter, the broadband impedance matching problem is formulated as a minimization problem of a pseudo hyperbolic distance in supremum norm over H^∞. The optimal point, if it exists, is obtained thanks to Nehari's theory which computes the supremum norm distance of an L^∞ function to H^∞ [4], [5]. This H^∞ approach guarantees the global optimality of the obtained response but at the cost of the absence of a degree constraint on the circuital response. The relative mathematical complexity of this approach together with the impossibility to realize, in practice, an infinite degree matching network limited its impact in electronics.

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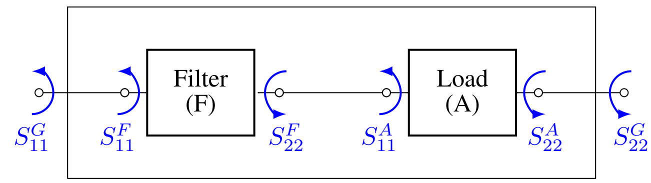

Fig. 1. Global system (Matching Filter + Load) and Scattering Parameters

II. THEORY

In this work, the matching problem is formulated as a convex optimization problem based on Youla's matching theory. The existence of finite degree matching networks that can achieve the optimal reflection level in a desired frequency band when plugged on to the given load is discussed. Given measured frequency data of the reflection coefficient of the load on the frequency band of interest, the first step is to obtain a rational approximation of S_{11}^A. Darlington's representation is utilized to construct an equivalent lossless two port, S^A with the same input reflection, S_{11}^A [2]. So, we have a load characterized by its 2×2 scattering matrix S^A while the matching filter to be synthesized is determined by its 2×2 scattering matrix S^F. Following the Fano-Youla matching theory, physical realizability of the global system (G) composed of the matching filter (F) chained to the Darlington equivalent of load (A) is first considered. Fig. 1 represents the cascade operation between F and A to form $G = F \circ A$. The minimization, over the pass-band, of a reflection coefficient associated to the global system, is cast to a convex optimisation problem involving Nevanlinna-Pick interpolation theory. The matching filter providing the optimal reflection level is afterwards obtained by de-embedding the load from the "optimal" global system.

In this work, $\Pi^+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ will be the domain of analyticity. The imaginary axis, $i\mathbb{R}$ is the frequency axis and the frequency variable as usual, $s = j\omega$. The elements of the scattering matrices are rational Schur functions (complex analytic functions bounded by one in modulus). For a rational matrix valued function, S, we define its para-conjugate, S^*, on the imaginary axis as,

$$S^*(s) = (\overline{S(-\overline{s})})^t \quad (1)$$

where superscript, ^{t}, stands for the transpose. The load (A) is supposed to be a system which does not completely reflect all frequencies, i.e $|S_{11}^A|$ is not uniformly equal to one for all frequencies and the filter is considered lossless, that is,

$$S^{F^*}(s)S^F(s) = Id. \tag{2}$$

Definition II.1. *(Chaining). The chaining of rational Schur function S_{22}^F, with the two port A, produces the output reflection coefficient S_{22}^G of the global system. We note $S_{22}^G = S_{22}^F \circ A$. The reflection coefficient S_{22}^G at each frequency, can be expressed as [6],*

$$\begin{aligned} S_{22}^G(s) &= S_{22}^A(s) + \frac{S_{12}^A(s)S_{21}^A(s)S_{22}^F(s)}{1 - S_{22}^F(s)S_{11}^A(s)} \qquad (3)\\ &= \frac{S_{22}^A(s) - S_{22}^F(s)det(S^A(s))}{1 - S_{22}^F(s)S_{11}^A(s)} \\ &= det(S^A(s))\left(\frac{S_{11}^{A^*}(s) - S_{22}^F(s)}{1 - S_{22}^F(s)S_{11}^A(s)}\right) \end{aligned}$$

Hence the modulus of the output reflection coefficient of the global system is obtained as the pseudo hyperbolic distance between S_{22}^F and $S_{11}^{A^*}$:

$$|S_{22}^G| = \left|\frac{S_{22}^F - S_{11}^{A^*}}{1 - S_{22}^F S_{11}^A}\right| = \delta\left(S_{22}^F, S_{11}^{A^*}\right) \tag{4}$$

A. De-embedding Approach

Definition II.2. - *(De-chaining). A lossless two port, A is said to be de-chainable of a rational Schur function, S_{22}^G if there exists a lossless two port F, with $S_{22}^F \in H^\infty(\Pi^+)$ (bounded holomorphic functions in the open right half plane), such that $S_{22}^G = S_{22}^F \circ A$. Inverting the operation mentioned in (3), we get the de-chaining formula,*

$$S_{22}^F(s) = \frac{S_{22}^G(s) - S_{22}^A(s)}{S_{22}^G(s)S_{11}^A(s) - det(S^A)} \tag{5}$$

Definition II.3. *(Transmission Zero). A transmission zero (possibly at ∞) associated to any lossless two port with scattering matrix S , can be defined as follows:*

$$Tz(S) = \{\lambda \in \overline{\Pi^+} : S_{12}S_{21}(\lambda) = 0\} \tag{6}$$

The $\lambda' s$ on the imaginary axis in the set $Tz(S)$ are counted for half their multiplicity.

Let $\mathbb{F}$ denote the set of all rational Schur functions, S_{22}^G such that the load A (lossless two port, with scattering matrix S^A) is de-chainable from S_{22}^G.

Theorem II.4. *(Characterisation of $\mathbb{F}$). Let A be a load characterized by its 2×2 lossless scattering matrix. Let $\{\xi_1, \xi_2, \dots, \xi_m\}$ be the m transmission zeros of A in Π^+ with multiplicity $M(\xi_k)$ for any general element ξ_k. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be its n transmission zeros on the imaginary axis with multiplicity $M(\alpha_k)$ for any general element α_k. For any function f, $f^{(0)} = f$ and $f^{(i)}$ represent the i-th derivative of f.*

The load A is de-chainable of S_{22}^G (i.e $S_{22}^G \in \mathbb{F}$) iff at each transmission zero of A, the following conditions are satisfied:

$$S_{22}^{G\,(i)}(\xi_k) = S_{22}^{A\,(i)}(\xi_k),\ 0 \le i \le M(\xi_k) - 1 \tag{7}$$

$$S_{22}^{G\,(i)}(\alpha_k) = S_{22}^{A\,(i)}(\alpha_k),\ 0 \le i \le 2M(\alpha_k) - 2 \tag{8}$$

For $i = 2M(\alpha_k) - 1$,

$$\overline{S_{22}^G(\alpha_k)}S_{22}^{G\,(i)}(\alpha_k) \ge \overline{S_{22}^A(\alpha_k)}S_{22}^{A\,(i)}(\alpha_k) \tag{9}$$

B. Convex Formulation

A general loss-less rational scattering matrix can be expressed in Belevitch form as follows [7] :

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \frac{1}{q}\begin{pmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{pmatrix} \tag{10}$$

where ϵ is a unimodular complex constant and p, q, r are complex polynomials of finite degree satisfying the Feldtkeller equation $qq^* = pp^* + rr^*$. In particular, the scattering matrix of the overall system G, can be put in Belevitch form. Its transmission polynomial r is shown to be equal to $r = r^F r^A$, the product of the transmission polynomials associated to the load and the matching filer respectively. We suppose that r^F is fixed by the user and let $N \ge deg(r^F r^A)$ be the target degree of our global system G. Let $P = pp^*$ and $R = rr^*$, where p (resp. r) is the reflection (resp. transmission) polynomial of the global system. On the imaginary axis we have:

$$|S_{22}^G|^2 = \frac{pp^*}{qq^*} = \frac{pp^*}{pp^* + rr^*} = \frac{1}{1 + \frac{R}{P}} \overset{\text{def}}{=} h(P)^2 \tag{11}$$

Let $\mathbb{P}_{2N}^+$ denote the positive polynomials of degree at most $2N$. We define the following set,

$$\mathbb{F}_R^N = \left\{f \in \mathbb{F} \mid \exists P \in \mathbb{P}_{2N}^+ : |f(s)| = h(P)\right\} \tag{12}$$

Let B be the desired passband, given as a finite union of compact intervals over the frequencies. We state our matching problem as,

Problem ($\mathfrak{P}$). *Find* $\quad l_{opt} = \min\limits_{S_{22}^G \in \mathbb{F}_R^N} \max\limits_{s \in B} |S_{22}^G(s)|$

We will derive a convex formulation of problem $\mathfrak{P}$, for which we need following notion of admissibility.

Definition II.5. *(Admissible Function). For a given load A, the function $h(P)$ is defined to be admissible if there exists a rational Schur function S_{22}^G, satisfying the following two conditions:*

- *The load A is de-chainable of S_{22}^G (i.e $S_{22}^G \in \mathbb{F}$).*
- *The modulus of output reflection coefficient, S_{22}^G formed by chaining S_{22}^F with A is bounded by $h(P)$ for all frequencies.*

$$|S_{22}^G(s)| = \delta(S_{22}^F(s), S_{11}^{A*}(s)) \le h(P)(s)$$

Using the above notion of admissibility, we can define a set $\mathbb{H}_R^N$ of positive polynomials as follows:

$$\mathbb{H}_R^N = \{P \in \mathbb{P}_{2N}^+ : h(P) \text{ is admissible}\} \tag{13}$$

Theorem II.6. *$\mathbb{H}_R^N$ is a convex set of positive polynomials.*

Proof. Let $P_1 \in \mathbb{H}_R^N$ and $P_2 \in \mathbb{H}_R^N$. By (13), $h(P_1)$ and $h(P_2)$ are admissible i.e there exists $S_{22}^{F_1}$ and $S_{22}^{F_2}$ in $H^\infty(\Pi^+)$ such that $S_{22}^{G_1} = S_{22}^{F_1} \circ A$, $S_{22}^{G_2} = S_{22}^{F_2} \circ A$ and at any s, we have,

$$|S_{22}^{G_1}| = \delta(S_{22}^{F_1}, S_{11}^{A*}) \leq h(P_1) \quad (14)$$

$$|S_{22}^{G_2}| = \delta(S_{22}^{F_2}, S_{11}^{A*}) \leq h(P_2) \quad (15)$$

$S_{22}^{G_1}$ and $S_{22}^{G_2}$ satisfies the conditions mentioned in Theorem II.4. Now, let $P_3 = \gamma P_1 + (1-\gamma)P_2$, for $0 < \gamma < 1$. We need to check whether, $P_3 \in \mathbb{H}_R^N$ i.e $h(P_3)$ is admissible. It can be easily verified that at every s of the imaginary axis, $h(P)(s)$ is a concave function of P. Note $x = P(s) \geq 0$ and $\alpha = R(s)$ and define,

$$h(P)(s) = \tilde{h}(x) \stackrel{\text{def}}{=} \sqrt{\frac{x}{x+\alpha}}. \quad (16)$$

Let us evaluate the first and second derivative of $\tilde{h}$ with respect to the non-negative real variable x,

$$\frac{d}{dx}(\tilde{h}(x)) = \frac{\alpha}{2}\frac{x^{\frac{-1}{2}}}{(x+\alpha)^{\frac{3}{2}}} \quad (17)$$

$$\frac{d^2}{dx^2}(\tilde{h}(x)) = \frac{-\alpha}{4}\left[\frac{(x+\alpha)^{\frac{3}{2}}x^{\frac{-3}{2}} + 3x^{\frac{-1}{2}}(x+\alpha)^{\frac{1}{2}}}{(x+\alpha)^3}\right] \leq 0 \quad (18)$$

showing that $\tilde{h}$ is a concave function in $x \geq 0$. It follows that at any s on the imaginary axis,

$$\begin{aligned} h(P_3)(s) &= \tilde{h}(\gamma P_1(s) + (1-\gamma)P_2(s)) \\ &\geq \gamma h(P_1)(s) + (1-\gamma)h(P_2)(s) \text{ (concavity of } \tilde{h}) \\ &\geq \gamma|S_{22}^{G_1}(s)| + (1-\gamma)|S_{22}^{G_2}(s)| \text{ (using(14), (15))} \\ &\geq |\gamma S_{22}^{G_1}(s) + (1-\gamma)S_{22}^{G_2}(s)| \text{ (Triangle Inequality)} \end{aligned}$$

So, we have an $S_{22}^{G_3} \stackrel{\text{def}}{=} \gamma S_{22}^{G_1} + (1-\gamma)S_{22}^{G_2}$ satisfying,

$$|S_{22}^{G_3}(s)| \leq h(P_3)(s) \quad (19)$$

Since $S_{22}^{G_3}$ is a linear combination of $S_{22}^{G_1}$ and $S_{22}^{G_2}$, it satisfies the de-embedding conditions in Theorem II.4 as well. So there exists $S_{22}^{F_3} \in H^\infty$ such that $S_{22}^{G_3} = S_{22}^{F_3} \circ A$ and at any s,

$$|S_{22}^{G_3}| = \delta(S_{22}^{F_3}, S_{11}^{A*}) \leq h(P_3) \quad (20)$$

Hence $h(P_3)$ is admissible and $P_3 \in \mathbb{H}_R^N$. □

Problem $\mathfrak{P}$ can be formulated on the convex set $\mathbb{H}_R^N$ as follows:

Problem ($\mathfrak{P}_C$)**.** *Find* $L_{opt} = \min_{P \in \mathbb{H}_R^N} \max_{s \in B} \frac{P}{R}(s)$

In this work, let us consider the case of a load having only m simple transmission zeros, $\{\xi_1, \xi_2, \ldots, \xi_m\}$, inside the analyticity domain (Π^+). Let U_P be defined as a rational Schur function of degree N which is outer (no roots inside Π^+) and satisfying,

$$|U_P(s)| = h(P)(s) = \sqrt{\frac{P(s)}{P(s)+R(s)}} \quad (21)$$

C. Characterisation of $\mathbb{H}_R^N$

A characterisation of the set $\mathbb{H}_R^N$ is obtained using Nevanlinna-Pick interpolation theory [5].

Proposition II.7. *The set* $\mathbb{H}_R^N$ *defined in* (13) *is characterised as,*

$$\tilde{\mathbb{H}}_R^N = \{P \in \mathbb{P}_{2N}^+ : \Delta(P) \succeq 0\}, \quad (22)$$

where the Pick matrix $\Delta(P) = [\Delta_{ij}]_{1 \leq i,j \leq m}$ *is defined by,*

$$\Delta_{ij} = \frac{1 - \overline{\left(\frac{S_{22}^A(\xi_i)}{U_P(\xi_i)}\right)}\left(\frac{S_{22}^A(\xi_j)}{U_P(\xi_j)}\right)}{\bar{\xi_i} + \xi_j} \quad (23)$$

Proof. We have,

$$\mathbb{H}_R^N = \{P \in \mathbb{P}_{2N}^+ : h(P) \text{ is admissible}\} \quad (24)$$

So, $\mathbb{H}_R^N$ consists of positive polynomials $P \in \mathbb{P}_{2N}^+$ for which there exists an $S_{22}^F \in H^\infty(\Pi^+)$ satisfying,

$$S_{22}^G = S_{22}^F \circ A, \text{ i.e } S_{22}^G(\xi_k) = S_{22}^A(\xi_k) \text{ (Theorem II.4)} \quad (25)$$

$$|S_{22}^G(s)| \leq h(P)(s) \quad (26)$$

Since, U_P is a rational outer function, we can divide (25) and (26) throughout by U_P and $|U_P|$ respectively. Now, the set $\mathbb{H}_R^N$ can be reformulated as set of positive polynomials $P \in \mathbb{P}_{2N}^+$ for which there exists $\frac{S_{22}^G}{U_P} \in H^\infty(\Pi^+)$ satisfying,

$$\left(\frac{S_{22}^G}{U_P}\right)(\xi_k) = \left(\frac{S_{22}^A}{U_P}\right)(\xi_k) \quad (27)$$

$$\left|\frac{S_{22}^G(s)}{U_P(s)}\right| \leq 1 \quad (28)$$

If we denote $\frac{S_{22}^G}{U_P} = S_P$, using classical Pick theorem, there exist a Schur function, $S_P : \Pi^+ \to \overline{\mathbb{D}}$ such that :

$$\forall k = \{1, 2, \ldots m\}, \; S_P(\xi_k) = \frac{S_{22}^A(\xi_k)}{U_P(\xi_k)} \quad (29)$$

iff the Pick matrix, $\Delta(P) = [\Delta_{ij}]_{1 \leq i,j \leq m}$ defined by,

$$\Delta_{ij} = \frac{1 - \overline{\left(\frac{S_{22}^A(\xi_i)}{U_P(\xi_i)}\right)}\left(\frac{S_{22}^A(\xi_j)}{U_P(\xi_j)}\right)}{\bar{\xi_i} + \xi_j}$$

is positive semidefinite. So, we have,

$$\mathbb{H}_R^N \subseteq \{P \in \mathbb{P}_{2N}^+ : \Delta(P) \succeq 0\} \stackrel{\text{def}}{=} \tilde{\mathbb{H}}_R^N$$

Now, we can prove the other inclusion, $\tilde{\mathbb{H}}_R^N \subseteq \mathbb{H}_R^N$. Consider a $P \in \tilde{\mathbb{H}}_R^N$. We have, $\Delta(P) \succeq 0$ and hence there exists a Schur function $S_P : \Pi^+ \to \overline{\mathbb{D}}$ such that :

$$\forall k = \{1, 2, \ldots m\}, \; S_P(\xi_k) = \frac{S_{22}^A(\xi_k)}{U_P(\xi_k)} \quad (30)$$

Now, we have a rational Schur function, $S_{22}^G = S_P U_P$, which verifies,

$$S_{22}^G(\xi_k) = S_{22}^A(\xi_k) \text{ (Using 30)} \quad (31)$$

$$|S_{22}^G(s)| \leq h(P)(s) \quad (32)$$

(32) follows, since S_P is a Schur function and $|U_P(s)| = h(P(s))$ (from (21)). This implies that $P \in \mathbb{H}_R^N$. So, we have $\tilde{\mathbb{H}}_R^N \subseteq \mathbb{H}_R^N$ and hence $\mathbb{H}_R^N = \tilde{\mathbb{H}}_R^N$. □

D. Properties of set $\mathbb{H}_R^N$

1) $\mathbb{H}_R^N$ is a non-empty set. This follows from the fact that, for a given load A with $|S_{22}^A| = h(P_A)$, we have $P_A \in \mathbb{H}_R^N$.
2) $\mathbb{H}_R^N$ is closed. The closure of $\mathbb{H}_R^N$ is obtained by proving that any convergent sequence of positive polynomials (P_n) in $\mathbb{H}_R^N$ converges to a positive polynomial $\tilde{P}$ such that $h(\tilde{P})$ is admissible. This is obtained by building a sequence of rational Schur functions $(S_{22}^{G_n})$ such that $|S_{22}^{G_n}| \leq h(P_n)$ and verifying the interpolation conditions of theorem II.4, and showing that a subsequence of the latter can be extracted that converges to S_{22}^G such that $|S_{22}^G| \leq h(\tilde{P})$.

E. Existence and Uniqueness Theorem

Theorem II.8. *The Problem $\mathfrak{P}_C$ defined in Section II.B is feasible and there exists a unique $P_{opt} \in \mathbb{H}_R^N$ at which L_{opt} is attained. Moreover $det(\Delta(P_{opt})) = 0$.*

Proof. Feasibility : The set $\mathbb{H}_R^N$ being non-empty yields the feasibility of Problem $\mathfrak{P}_C$.
Existence of P_{opt} : Let us denote the cost function as $\Psi(P) = \max_{s \in B} \frac{P}{R}(s)$. Consider a minimizing sequence, $(P_n) \in \mathbb{H}_R^N$, such that $\lim_{n\to\infty} (\Psi(P_n)) = \inf_{P \in \mathbb{H}_R^N} \Psi(P)$. The sequence (P_n) is trivially bounded, hence we can extract a subsequence converging to an element P_{opt} in $\mathbb{H}_R^N$ and such that $\Psi(P_{opt}) = L_{opt}$.
Uniqueness of P_{opt} : Using the strictly contractive nature of particular Pick interpolants corresponding to positive definite matrices, we first prove that $\Delta(P_{opt})$ is singular at any P_{opt} such that $\Psi(P_{opt}) = L_{opt}$. Reformulating the characterisation of $\mathbb{H}_R^N$, using H^∞'s classical extremal approximation problem, we consider:

$$\min_{h \in H^\infty} ||S_{22}^A U_P^{-1} B^{-1} - h||_\infty \tag{33}$$

where B denote the finite Blaschke product whose zeros are precisely at ξ_k. For any $P_{opt} \in \mathbb{H}_R^N$, at which L_{opt} is attained, the minimum in (33) is equal to one. Suppose now that there are two distinct optimal solutions, P_1 and P_2 in $\mathbb{H}_R^N$: an argument based on the circular nature of the error function in problem (33) shows that $det(\Delta((P_1+P_2)/2)) > 0$, hence a contradiction. □

III. RESULTS

The solution to problem $\mathfrak{P}_C$ will thus provide a $P_{opt} \in \mathbb{H}_R^N$ and the associated admissible function $h(P_{opt})$ will yield a rational Schur function, $S_{22}^G = U_{P_{opt}} S_{P_{opt}}$. Pick matrix, $\Delta(P_{opt})$ being singular implies $S_{P_{opt}}$ is a Blaschke product and so, $S_{22}^G \in \mathbb{F}_R^N$ solves problem $\mathfrak{P}$. This in turn provides an $S_{22}^F \in H^\infty(\Pi^+)$ of finite degree which will furnish the synthesis of finite degree matching network F that provides the optimal reflection level when plugged on to the load. We present a practical example of matching filter synthesis for a given super directive antenna used for mobile communication in the passband 870 MHz to 890 MHz.

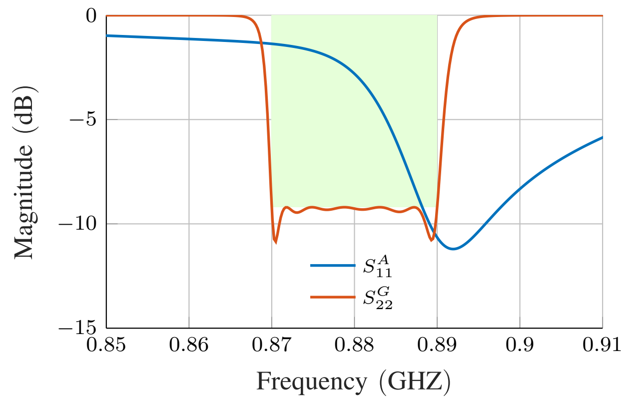

Fig. 2. Result of Problem $\mathfrak{P}_C$ (load of degree 2 and system of degree 6).

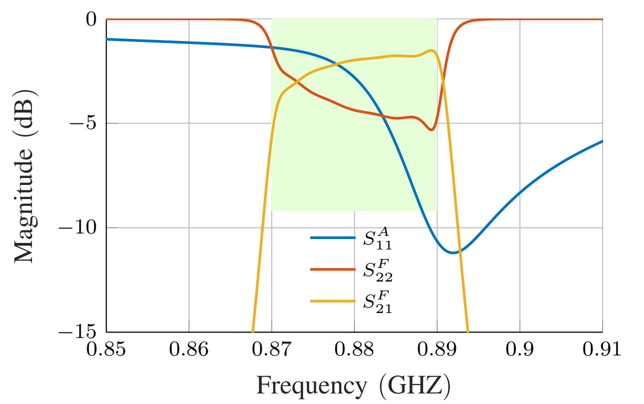

Fig. 3. Matching filter providing the response in Fig. 2.

Fig. 2 shows the reflection of antenna (S_{11}^A) along with the global system reflection (S_{22}^G) that is obtained by solving Problem $\mathfrak{P}$ (taking $N = 6$). By using the matching filter, the reflection at the left edge of the band (870MHz) has been improved from -1.3dB to nearly -9dB. Parameters S_{22}^F and S_{21}^F of the matching filter that provides this result are shown in Fig. 3, together with the load reflection S_{11}^A.

The practical implementation is performed using optimization techniques based on linear matrix inequalities (positivity of polynomial P) and non-linear, yet convex, matrix inequalities relative to the involved Pick matrix.

REFERENCES

[1] R.M. Fano, Theoretical Limitations on the Broadband Matching of Arbitrary Impedances, Technical Report Volume 249, MIT, 1950.
[2] H.J. Carlin and P.P. Civalleri, Wideband Circuit Design, CRC Press, 1997.
[3] J. William Helton, Non-Euclidean Functional Analysis and Electronics, Bull. Amer. Math. Soc. (N.S.), 7(1):1-64, 07 1982.
[4] Nicholas Young, An Introduction to Hilbert Spaces, Cambridge University Press, 1988.
[5] John B. Garnett, Bounded Analytic Functions, Springer, 2007.
[6] L. Baratchart, M. Olivi, and F. Seyfert, Boundary Nevanlinna–Pick Interpolation with Prescribed Peak Points. Application to Impedance Matching, SIAM Journal on Mathematical Analysis, 2017, Vol. 49, pp. 1131-1165.
[7] V. Belevitch, Classical Network Theory, S. Francisco, Holden-Day, 1968.
[8] B. Dumitrescu, Positive Trigonometric Polynomials and Signal Processing Applications. Springer, Jan. 2007.
[9] M. Stingl, On the solution of nonlinear semidefinite programs by augmented Lagrangian methods. Friedrich Alexander University Nuremberg, 2006.