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Strong input-to-state stability for infinite dimensional linear systems

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***Abstract*— In this talk we study the notions of strong input-to-state stability and strong integral input-to-state stability in the setting of linear systems with an unbounded control operator. It is known that if the system is exponentially stable, then it is (strong) integral input-to-state stable if and only if it is infinite-time admissible with respect to inputs in an Orlicz space. Without the exponential stability those conditions are no longer equivalent. Still, the Orlicz space inifnite-time admissibility is sufficient for a system to be strong integral input-to-state stable.**

***Index Terms*— input-to-state stability, integral input-to-state stability, infinite-dimensional systems**

I. Introduction

The *input-to-state stability* (ISS) for PDE systems is a comparatively new research topic. It started with [1] and we also refer to [2], [3], [4], [5], [6], [7], [8] for the current state of research in this area. One of the questions people study is how ISS is related to other stability notions such as *integral input-to-state stability* (iISS). In infinite dimensional setting the situation is not fully clear, not even for linear systems, see [9].

In this talk we present results recently obtained in [10]. Let us consider linear control systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ and B is a (possibly unbounded) control operator. For linear systems given by (1) ISS allows to jointly describe the stability of the semigroup $(T(t))_{t\geq 0}$ together with the stability of the map $u \mapsto x(t)$ with respect to some space Z of input functions, for fixed $t > 0$. The more general notion of *strong ISS* (sISS) was recently introduced in [11], see also [8]. It generalizes ISS as now the exponential stability of the semigroup is replaced with strong stability.

One of the main results in [9] is the equivalence between integral ISS with respect to L^∞ and ISS with respect to some Orlicz space E_Φ. As the system is linear, E_Φ-ISS is equivalent to admissibility with respect to E_Φ. In contrast to this L^∞-siISS is implied by E_Φ-sISS but not the other way around.

The equivalence fails to hold true since, unlike in the relation between ISS and iISS, we have to distinguish between "finite-time admissibility" and "infinite-time admissibility", as the latter is strictly stronger here.

II. Definitions

We study systems $\Sigma(A,B)$ given by (1) where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, U is another Banach space and $B \in \mathcal{L}(U, X_{-1})$. The space X_{-1} is defined to be the completion of X with respect to the norm given by $\|x\|_{-1} := \|(\lambda I - A)^{-1}x\|$, where λ is some element of $\rho(A)$, the resolvent set of A. The operator A has a unique extension $A_{-1} \in \mathcal{L}(X, X_{-1})$ which generates a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} which is an extension of $(T(t))_{t\geq 0}$.

We recall the definitions of Young functions and Orlicz spaces. The Orlicz spaces generalize the usual L^p spaces. A function $\Phi\colon [0,\infty) \to \mathbb{R}$ is called a *Young function* if

$$\Phi(t) = \int_0^t \varphi(s)\, ds, \qquad t \geq 0,$$

where the function $\varphi\colon [0,\infty) \to \mathbb{R}$ has the following properties: φ is right-continuous and nondecreasing, $\varphi(0) = 0$, $\varphi(s) > 0$ for $s > 0$ and $\lim_{s\to\infty} \varphi(s) = \infty$. We denote by $\mathcal{K}_Y$ the set of all Young functions.

Definition 2.1: Let $I \subset \mathbb{R}$ be a bounded interval and $\Phi \in \mathcal{K}_Y$. The *Orlicz space* $E_\Phi(I,U)$ is the completion of $L^\infty(I,U)$ with respect to the *Luxemburg norm*

$$\|u\|_\Phi := \inf\left\{ k > 0 \;\middle|\; \int_I \Phi(k^{-1}\|u(x)\|_U)\, dx \leq 1 \right\}.$$

Let $p \in (1,\infty)$. Then taking $\Phi(s) = s^p$ yields $E_\Phi = L^p$, i.e. L^p spaces are special Orlicz spaces. More details can be found in [12] and also in the appendix of [9]. We use the following convention: By $Z(0,t;U)$ we refer to either a Lebesgue space $L^p(0,t;U)$, with $1 \leq p \leq \infty$ or an Orlicz spaces $E_\Phi(0,t;U)$, for some $\Phi \in \mathcal{K}_Y$.

By a *(mild) solution* of (1) we mean the function defined by the variation of parameters formula

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\, ds, \qquad t \geq 0. \quad (2)$$

Note that in general we have $x(t) \in X_{-1}$.

Definition 2.2: We call the system $\Sigma(A,B)$

- *Z-admissible*, if for all $t > 0$, $x_0 \in X$ and all $u \in Z(0,t;U)$ it holds that $x(t) \in X$ and there exists a constant $c(t)$ such that

$$\|x(t)\| \leq c(t)\|u\|_{Z(0,t;U)}. \quad (3)$$

  for all $u \in Z(0,t;U)$, if $x_0 = 0$.

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- *infinite-time Z-admissible*, if the system is Z-admissible and the optimal constants in (3) satisfy $c_\infty := \sup_{t>0} c(t) < \infty$.

In order to introduce (strong) input-to-state stability, we use the following notation for the sets of comparison functions.

$$\begin{aligned}
\mathcal{K} &= \{\mu \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) \mid \mu(0) = 0,\ \mu \text{ strictly increasing}\},\\
\mathcal{K}_\infty &= \{\theta \in \mathcal{K} \mid \lim_{x\to\infty} \theta(x) = \infty\},\\
\mathcal{L} &= \{\gamma \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) \mid \gamma \text{ str. decreas.}, \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} &= \{\beta\colon (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+ \mid \beta(\cdot, t) \in \mathcal{K}\ \forall t, \beta(s, \cdot) \in \mathcal{L}\ \forall s\}.
\end{aligned}$$

Cleary we have

$$\mathcal{K}_Y \subsetneq \mathcal{K}_\infty \subsetneq \mathcal{K}.$$

Definition 2.3: The system $\Sigma(A, B)$ is called

- *Z-ISS*, if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that for all $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$ holds $x(t) \in X$ and

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \mu(\|u\|_{Z(0,t;U)}).$$

- *Z-iISS*, if there exist $\beta \in \mathcal{KL}$, $\theta \in \mathcal{K}_\infty$ and $\mu \in \mathcal{K}$ such that for all $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$ holds $x(t) \in X$ and

$$\|x(t)\| \leq \beta(\|x_0\|, t) + \theta\left(\int_0^t \mu(\|u(s)\|_U) ds\right).$$

The relation between integral ISS with respect to L^∞ and ISS with respect to E_Φ is one of the main results in [9].

Theorem 2.4 ([9, Thm. 3.16]): Let $(T(t))_{t\geq 0}$ be exponentially stable. The following statements are equivalent.

1) $\Sigma(A, B)$ is L^∞-iISS.
2) There is a $\Phi \in \mathcal{K}_Y$ such that $\Sigma(A, B)$ is E_Φ-ISS.

Next we introduce the strong versions of ISS and iISS.

Definition 2.5: The system $\Sigma(A, B)$ is called

- *Z-sISS*, if there exist $\mu \in \mathcal{K}$ and $\beta\colon X \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that
  1) $\beta(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$ and
  2) for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$ holds $x(t) \in X$ and

$$\|x(t)\| \leq \beta(x_0, t) + \mu(\|u\|_{Z(0,t;U)}).$$

- *Z-siISS*, if there exist $\theta \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and $\beta\colon X \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that
  1) $\beta(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$ and
  2) for every $t \geq 0$, $x_0 \in X$ and $u \in Z(0, t; U)$ holds $x(t) \in X$ and

$$\|x(t)\| \leq \beta(x_0, t) + \theta\left(\int_0^t \mu(\|u(s)\|_U) ds\right). \quad (4)$$

It is easy to see that ISS implies sISS and iISS implies siISS.

III. Main Results

We relate L^∞-siISS to infinite-time admissibility with respect to Orlicz spaces E_Φ.

Theorem 3.1: Suppose there is a $\Phi \in \mathcal{K}_Y$ such that the system $\Sigma(A, B)$ is E_Φ-sISS. Then the system $\Sigma(A, B)$ is L^∞-siISS.

Theorem 3.2: Assume that the system $\Sigma(A, B)$ is L^∞-siISS. Then there is a $\Phi \in \mathcal{K}_Y$ such that the system $\Sigma(A, B)$ is E_Φ-admissible. If, additionally, for the function μ in (4) holds $\mu \in \mathcal{K}_Y$, then the system $\Sigma(A, B)$ is E_μ-sISS.

Next we see that E_Φ-sISS and L^∞-siISS are not equivalent, i.e. we cannot drop the additional condition in the second part of Theorem 3.2. Note that, in contrast to Theorem 2.4, Theorem 3.1 and the next one show that without exponential stability, E_Φ-sISS is stronger than L^∞-siISS.

Theorem 3.3: Let $(T(t))_{t\geq 0}$ be the left-shift semigroup on $X = L^1(0, \infty)$. Its generator is the operator $Af := f'$ with the domain $D(A) = \{f \in L^1(0, \infty) \mid f \in W^{1,1}(0, \infty) \text{ and } f' \in L^1(0, \infty)\}$. Choosing $U = X$ and $B = I$ we have:

1) The semigroup $(T(t))_{t\geq 0}$ strongly stable,
2) $\Sigma(A, B)$ is L^1-siISS and hence L^∞-siISS,
3) $\Sigma(A, B)$ is not E_Φ-sISS for any $\Phi \in \mathcal{K}_Y$.

Moreover $\Sigma(A, B)$ is not infinite-time L^∞-admissible. In particular L^∞-siISS does not imply L^∞-sISS.

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