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Linear Quadratic Optimal Control of Jump System over Multiple Erasure Channels

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***Abstract*— This paper considers the design of linear quadratic optimal controller for Markovian jump linear systems (MJLSs) over multiple lossy communication channels with correlated packet loss. Considering the Gilbert-Elliott channel model which takes into account the temporal correlation of the data packet losses, the finite horizon controller and the infinite horizon controller are designed under TCP-like protocol. Existence of the infinite horizon controller is investigated by analyzing convergence of the cost function. Further, we derive the finite horizon and the infinite horizon controller for a linear time invariant (LTI) system as a special case. Finally, for the infinite horizon case, we demonstrate the dependence of the convergence of the optimal cost on the control packet arrival probability.**

I. INTRODUCTION

Control over communication networks or Networked Control Systems (NCSs) have attracted considerable attention recently, owing to their numerous applications [1]–[3]. These are systems where different subsystems, i.e., controllers, actuators and sensors are spatially distributed and are connected through networks.

Introduction of communication networks leads to some serious issues, such as data packet loss [4] and random time delay [5]. Packet loss is generally modeled as independent and identically distributed (i.i.d.) Bernoulli process for the sake of mathematical simplicity [4]. However, realistic communication channels possess memory and i.i.d. Bernoulli model does not capture this [6]. In order to represent the temporal correlation in a channel, one can instead use the Gilbert-Elliott channel model [6], [7]. This model is based on a two state Markov chain. Depending on the availability of acknowledgement of data packet reception, there are two different types of protocols used by communication networks: TCP-like protocols and UDP-like protocols [4]. For a TCP-like protocol, there is an acknowledgement mechanism to know whether a packet is received or not. On the contrary, in a UDP-like protocol there is no such acknowledgement available.

The design of the Linear Quadratic Gaussian (LQG) controller for an NCS using a TCP-like protocol has been studied in [4] & [8]. In [8], considering Bernoulli packet loss, it has been shown that there exists a critical probability for packet arrival below which the closed loop system can not be stabilized. The LQG control problem for a UDP-like protocol has also been considered in [4], wherein it has been shown that while the separation principle holds true for a TCP-like, it fails for a UDP-like one. [9] deals with the design of the LQG controller for multi-input multi-output (MIMO) systems with Bernoulli packet loss. It is proved that the separation principle still holds true for the multi-channel case, when the protocol is TCP-like. In [6], the LQG control problem with Markovian packet losses has been considered. In [10], it has been shown that for a scalar system with quantization error, packet loss and computation delay, the estimator and the controller can not be designed independently. The LQG controller with correlated packet loss is designed in [11]. Considering consecutive packet losses, sub-optimal controllers are designed to reduce the complexity of the optimal controller.

In this paper, we investigate the optimal linear quadratic control problem of a Markovian jump linear system (MJLS) over multiple channels. An MJLS consists of a set of linear subsystems, where the system dynamics switches between the dynamics of the different subsystems according to a Markov chain. In many practical systems, such as robotic manipulators, aircraft systems, economic systems etc., the system structure changes due to various random factors, e.g., unforeseen environmental changes, random component failure, sudden change in operating point of the nonlinear system etc. and can be modeled as MJLS [12]. This work focuses on the control packet erasures and assumes that the sensor-to-controller channels are loss-less. We consider the case, where each of the communication channels have memory and model the channels using the Gilbert-Elliott channel model, wherein, packet losses are governed by a two state Markov chain. Here, the zero input strategy is followed, i.e., when the actuators do not receive control packets from the control unit, then the control input is zero. We concentrate on the TCP-like case, i.e., it is possible to know whether a packet sent from the controller has been received by the actuator or not.

The paper is organized as follows. In Section II., the problem is formulated along with an introduction to the Gilbert Elliott channel model. Section III. deals with design of the Finite horizon controller and the Infinite horizon controller. Convergence of the infinite horizon cost has also been investigated. In section IV. with a numerical example we demonstrate our results. Finally, section V. concludes the paper.

Notation: **diag**$\{a_1, ..., a_n\}$ represents diagonal matrix with $a_1, ..., a_n$ as diagonal elements, $||x|| := (x^T x)^{1/2}$ denotes the Euclidean norm, $||x||_P := (x^T P x)^{1/2}$ is the

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weighted Euclidean norm, $\rho(A)$ represent spectral radius of operator A. $\mathcal{L}_2([0,\infty),\mathbb{R}^n)$ is the space of square-summable functions x_k from $\mathbb{R}^n$ with $k = 0,1,...,\infty$. For a matrix P, $P \geq 0$ and $P > 0$ implies P is positive semi-definite and positive definite, respectively. The power set, i.e., set of all the subsets of set $\mathscr{I}$ is denoted by $2^{\mathscr{I}}$. $I_{m\times m}$ represents the identity matrix of order m.

II. Problem Formulation

Let us consider the following discrete-time Markovian jumped linear system:

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k^a \tag{1}$$

where $x_k \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, $u_k^a \in \mathcal{U} \subseteq \mathbb{R}^m$ is the control input applied by the actuator, $r_k \in \mathcal{D} \triangleq \{1,2,...,\mathcal{M}\}$ with $\mathcal{M} < \infty$ is a irreducible and homogeneous Markov chain. Transition probabilities of the Markov chain are expressed in the transition probability matrix $\Lambda = [p_{ij}]$, where

$$p_{ij} = P(r_{k+1} = j | r_k = i); \forall i,j \in \mathcal{D}, k = 0,1,2,...$$

Throughout the paper, it is assumed that state of the system x_k and Markov chain state r_k are directly accessible to the controller.
Let $u_k \in \mathbb{R}^m$ be the controller output, which has been sent to the actuators through lossy channels. Following expression relates u_k and u_k^a :

$$u_k^a = \xi_k u_k \tag{2}$$

where, $\xi_k = \text{diag}\{v_{k,1}, v_{k,2}, ..., v_{k,m}\}$ represents the packet loss condition of all the channels in the controller-to-actuator path.
Here, $v_{k,i}(i \in \{1,2,..,m\})$ are random variables, which correspond to packet loss condition in the i^{th} channel. $v_{k,i}$ is such that $v_{k,i} = 0$ implies that a packet is lost and $v_{k,i} = 1$ implying a successfull packet delivery. Here, we consider the case, where packet loss in each channel can be temporally correlated, which can modeled by the Gilbert-Elliott channel model. This model is based on a two-state Markov chain, where one state represents good state, i.e., successfull packet arrival and the other one represents bad state, i.e., packet loss [6], [7].

Now, let us define : $\bar{v}_l = P(v_{k,l} = 1 \big| v_{k-1,l} = 1)$, $\bar{\mu}_l = P(v_{k,l} = 1 \big| v_{k-1,l} = 0)$. So, the transition probabilities of the Markov states can be written by the following transition probability matrix:

$$\begin{aligned}&\begin{bmatrix} P(v_{k+1,i} = 0 | v_{k,i} = 0) & P(v_{k+1,i} = 1 | v_{k,i} = 0) \\ P(v_{k+1,i} = 0 | v_{k,i} = 1) & P(v_{k+1,i} = 1 | v_{k,i} = 1)\end{bmatrix} \\ &\qquad = \begin{bmatrix} 1-\bar{\mu}_i & \bar{\mu}_i \\ 1-\bar{v}_i & \bar{v}_i \end{bmatrix}\end{aligned} \tag{3}$$

Note 1. For the i^{th} channel, if there is no information available regarding the past packet loss conditions $(v_{j,i}; j < k)$, then the probability of a data packet loss (or, successful packet arrival) at any instant k will always be same and can be written as [6]:

$$\begin{aligned} P\big(v_{k,i} = 0\big) &= (1-\bar{v}_i)/(1+\bar{\mu}_i - \bar{v}_i) \\ P\big(v_{k,i} = 1\big) &= \bar{\mu}_i/(1+\bar{\mu}_i - \bar{v}_i) \end{aligned} \tag{4}$$

Note 2. It should be clearly noted that $\{r_k\}$ and $\{v_{k,i}\}$, $\forall i \in \{1,2,...,m\}$ are independent Markov processes.

Let us now define a few parameters which will be used in the following section. Let all the actuators are indexed by the set $\mathscr{G} = \{1,2,...,m\}$. Subsets $\mathscr{I}_i \subseteq \mathscr{G}$, $\forall i \in \{0,1,2,...,(2^m-1)\}$ are used to indicate which actuators have successfully received the control command. For $0 \leq i \leq 2^m - 1$, $\mathscr{I}_i \subseteq \mathscr{G}$ is defined as follows:

$$\mathscr{I}_i = \Big\{p \in \mathscr{G} : \ p^{th} \text{ entry in the } m\text{-bit binary representation of } i \text{ is } 1, \text{ where position number of the Least Significant Bit (LSB) is assumed to be } 1\Big\}$$

The nonzero entries of $\mathscr{I}_i$; $\forall i \in \{0,1,...,(2^m-1)\}$ correspond to those actuators that successfully receive control command. For example, $\mathscr{I}_3 = \{1,2\}$ indicates that 1^{st} and 2^{nd} actuators receive control command.
Following are a few related definitions:

(a) $\mathcal{N}(i) = \textbf{diag}\big\{a_{jj}\big\}$; where $a_{jj} = \begin{cases} 1, & \text{if } j \in \mathscr{I}_i \\ 0, & \text{if } j \notin \mathscr{I}_i \end{cases}$ for $j = 1,2,...,m$

(b) $\mathcal{P}_l^i = \prod_{j\in\mathscr{I}_l} P\big(v_{k,j} = 1 \big| \xi_{k-1} = \mathcal{N}(i)\big) \prod_{j\notin\mathscr{I}_l} P\big(v_{k,j} = 0 \big| \xi_{k-1} = \mathcal{N}(i)\big)$

(c) Let $\mathbf{Y}_k(.)$ be a map from $2^{\mathscr{G}}$ to a space which is closed under addition. Now, $\mathbf{L}^i(.)$ can be defined as follows:

$$\mathbf{L}^i\big(\mathbf{Y}_{k+1}(.)\big) = \sum_{l=0}^{2^m-1} \mathcal{P}_l^i \mathbf{Y}_{k+1}(l) \tag{5}$$

In the absence of any past information regarding packet loss status, $\mathcal{P}_l^i$ takes the following form (using *Note 1.*): $\hat{\mathcal{P}}_l = \prod_{j\in\mathscr{I}_l} P(v_{k,j} = 1) \prod_{j\notin\mathscr{I}_l} P(v_{k,j} = 0)$.

Remark 1. For the single-channel case, ξ_k can only be equal to $0_{m\times m}$ (when a packet gets lost in the channel) or $1_{m\times m}$ (when a packet is successfully received), i.e., partial information loss is not possible. However, in multi-channel case, as all the channels are independent, partial information loss is possible.

Now, for the TCP-like protocol, the information set $\mathcal{I}_k$ available to the controller at the k^{th} time-index can be defined as follows:

$$\mathcal{I}_k = \{x_0,...,x_k, r_0,...,r_k, \xi_0, ...\xi_{k-1}\} \tag{6}$$

Let us define the control policy ζ as follows: $\zeta = \{\zeta_0,...,\zeta_k,...\}$, where $\zeta_k : \mathcal{I}_k \rightarrow \mathcal{U}$ maps the information set $\mathcal{I}_k$ to some control input $\mathcal{U}$.

The rest of the paper focuses on finding control policy ζ, such that $u_k = \zeta_k(\mathcal{I}_k)$ minimizes the following cost function:

$$\begin{aligned} J_N &= \mathbb{E}\Big[||x_N||^2_{W_N} + \sum_{k=0}^{N-1} ||x_k||^2_{W_k} + ||u^a_k||^2_{R_k}\Big] \\ &= \mathbb{E}\Big[||x_N||^2_{W_N} + \sum_{k=0}^{N-1} ||x_k||^2_{W_k} + ||\xi_k u_k||^2_{R_k}\Big] \end{aligned} \tag{7}$$

where W_k and R_k are symmetric matrices such that $W_k \geq 0$ and $R_k > 0$ for all k.

III. Main Results

In this section, using dynamic programming, we shall find the optimal control law and the optimal value of the cost function given in (7).

A. Finite horizon control:

Let us write the value function, i.e., the cost-to-go from k^{th} stage as follows:

$$V_{k,N}(x_k, \xi_{k-1}, r_k) = \min_{u_k} \mathbb{E}\Big[||x_N||^2_{W_N} + \sum_{j=k}^{N-1} ||x_j||^2_{W_j} + ||\xi_j u_j||^2_{R_j}\Big]$$

Now, using Bellman's principle of optimality, we can write the value function as follows:

$$\begin{aligned} V_{k,N}(x_k, \xi_{k-1}, r_k) = \min_{u_k} \mathbb{E}\ \Big[& x_k^T W_k x_k + u_k^T \xi_k^T R_k \xi_k u_k \\ &+ V_{k+1,N}(x_{k+1}, \xi_k, r_{k+1}) | \mathcal{I}_k \Big] \end{aligned} \tag{8}$$

Lemma 1. For the cost function (7), subjected to system dynamics (1), the following claims are true.

(a) Suppose at the $(k-1)^{th}$ time index, packet loss status in the controller-to-actuator path is $\xi_{k-1} = \mathcal{N}(i);\ \ \forall i \in \{0, 1, 2, ..., (2^m-1)\}$ and $r_k = j \in \mathcal{D}$, then the value function at stage $k \in [0, N]$ can expressed as follows:

$$V_{k,N}\Big(x_k, \mathcal{N}(i), j\Big) = x_k^T \Xi_{k,N}(i,j) x_k \tag{9}$$

where $\Xi_{k,N}(i,j)$ is a symmetric matrix and is generated by the following Coupled Algebraic Riccati Equations (CAREs):

$$\begin{aligned} &\Xi_{k,N}(i,j) = W_k + A^T(j)\mathbf{L}^i\Big(\mathbf{X}_{k+1,N}(.,j)\Big)A(j) \\ &- A^T(j)\Big[\mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\mathbf{X}_{k+1,N}(.,j)\Big)\Big]^T \\ &\times \Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R_k + B^T(j)\mathbf{X}_{k+1,N}(.,j)B(j)\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\ &\times \mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\mathbf{X}_{k+1,N}(.,j)\Big)A(j) \end{aligned} \tag{10}$$

$\mathbf{L}^i(.)$ is defined in (5) and $\mathbf{X}_{k+1,N}(l,j)$ in (5) is defined as follows:

$$\mathbf{X}_{k+1,N}(l,j) = \sum_{t=1}^{\mathcal{M}} \Big(p_{jt}\Xi_{k+1,N}(l,t)\Big) \tag{11}$$

(b) The optimal control law is given by

$$\begin{aligned} u_k^* = -\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R_k + B^T(j)\mathbf{X}_{k+1,N}(.,j)B(j)\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\ \times \mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\mathbf{X}_{k+1,N}(.,j)\Big)A(j)x_k \end{aligned} \tag{12}$$

(c) The optimal cost for finite horizon is given by

$$J_N^* = \sum_{l=0}^{2^m-1} \hat{\mathcal{P}}_l \Big\{ x_0^T \Xi_{0,N}(l, r_0) x_0 \Big\}$$

Proof: We prove the *Lemma* using induction.
For the stage $k = N$, it is trivial to see that $V_{N,N}\Big(x_N, \mathcal{N}(i), j\Big) = \mathbb{E}\Big[x_N^T \Xi_{N,N}(i,j) x_N \Big| \mathcal{I}_N\Big]$ where $\Xi_{N,N}(i,j) = W_N$, $\forall i \in \{0, 1, ...(2^m-1)\}$ and $\forall j \in \mathcal{D}$. Let us now assume that claim (a) is true for the $(k+1)^{th}$ stage. So, with information set $\mathcal{I}_{k+1}$, we can represent $V_{k+1,N}(x_{k+1}, \xi_k, r_{k+1})$ as follows:

$$\begin{aligned} V_{k+1,N}\Big(x_{k+1}, \mathcal{N}(i), j\Big) &= x_{k+1}^T \Xi_{k+1,N}(i,j) x_{k+1}; \\ &\text{if } \ \xi_k = \mathcal{N}(i) \ \ \& \ \ r_{k+1} = j \end{aligned} \tag{13}$$

Now, with information set $\mathcal{I}_k$:

$$\begin{aligned} &\mathbb{E}\big[V_{k+1,N}(x_{k+1}, \xi_k, r_{k+1}) \big| \mathcal{I}_k\big] \\ &= \sum_{l=0}^{2^m-1} \mathcal{P}_l^i \Big\{ \Big(A(j)x_k + B(j)\mathcal{N}(l)u_k\Big)^T \sum_{t=1}^{\mathcal{M}} \big(p_{jt}\Xi_{k+1,N}(l,t)\big) \\ &\times \Big(A(j)x_k + B(j)\mathcal{N}(l)u_k\Big)\Big\} \\ &= \sum_{l=0}^{2^m-1} \mathcal{P}_l^i \Big\{ x_k^T A^T(j) \mathbf{X}_{k+1,N}(l,j) A(j) x_k \\ &+ u_k^T \mathcal{N}(l) B^T(j) \mathbf{X}_{k+1,N}(l,j) B(j) \mathcal{N}(l) u_k \\ &+ 2u_k^T \mathcal{N}(l) B^T(j) \mathbf{X}_{k+1,N}(l,j) A(j) x_k \Big\} \end{aligned} \tag{14}$$

Combining (8) and (14):

$$\begin{aligned} &V_{k,N}\Big(x_k, \mathcal{N}(i), j\Big) \\ &= \min_{u_k} \Big\{ \ x_k^T W_k x_k + x_k^T A^T(j) \mathbf{L}^i\Big(\mathbf{X}_{k+1,N}(.,j)\Big) A(j) x_k \\ &+ u_k^T \mathbf{L}^i\Big(\mathcal{N}(.)\Big(R_k + B^T(j)\mathbf{X}_{k+1,N}(.,j)B(j)\Big)\mathcal{N}(.)\Big) u_k \\ &+ 2u_k^T \mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\mathbf{X}_{k+1,N}(.,j)\Big)A(j)x_k \Big\} \end{aligned} \tag{15}$$

From the above equation, optimal control law can be derived as follows:

$$\begin{aligned} u_k^* = -\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R_k + B^T(j)\mathbf{X}_{k+1,N}(.,j)B(j)\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\ \times \mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\mathbf{X}_{k+1,N}(.,j)\Big)A(j)x_k \end{aligned} \tag{16}$$

Substituting the optimal control law back in equation (15), we get:

$$\begin{aligned} &V_{k,N}\Big(x_k, \mathcal{N}(i), j\Big) \\ &= x_k^T W_k x_k + x_k^T A^T(j) \mathbf{L}^i\Big(\mathbf{X}_{k+1,N}(.,j)\Big) A(j) x_k \\ &- x_k^T A^T(j) \Big(\mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\mathbf{X}_{k+1,N}(.,j)\Big)\Big)^T \\ &\times \Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R_k + B^T(j)\mathbf{X}_{k+1,N}(.,j)B(j)\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\ &\times \mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\mathbf{X}_{k+1,N}(.,j)\Big)A(j)x_k \end{aligned}$$

Hence,

$$V_{k,N}\Big(x_k, \mathcal{N}(i), j\Big) = x_k^T \Xi_{k,N}(i,j) x_k \tag{17}$$

where $\Xi_{k,N}(i,j)$ is given by CAREs (10).
The optimal value of the cost (7) will be the expected value of the value function at stage $k = 0$, i.e., $\mathbb{E}\Big[V_{0,N}(x_0,\xi_{-1},r_0)\Big|\mathcal{I}_0\Big]$. So,

$$\begin{aligned} J_N^* &= \mathbb{E}\Big[V_{0,N}(x_0,\xi_{-1},r_0)\Big|\mathcal{I}_0\Big] \\ &= \sum_{l=0}^{2^m-1} \hat{\mathcal{P}}_l\Big\{x_0^T\Xi_{0,N}(l,r_0)x_0\Big\} \end{aligned} \tag{18}$$

□

Remark 2. The finite horizon linear quadratic optimal controller for a linear time invariant (LTI) system can be derived from *Lemma 1* by considering the special case $\mathcal{D} = \{1\}$. The optimal controller can be expressed as follows:

$$\begin{aligned} u_k^* = -\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R_k + B^T\Xi_{k+1,N}(.)B\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\ \times\mathbf{L}^i\Big(\mathcal{N}(.)B^T\Xi_{k+1,N}(.)\Big)Ax_k \end{aligned}$$

where, $A(1) = A$, $B(1) = B$ and $\Xi_{k,N}(i)$ for $0 \le i \le 2^m-1$ is generated by the following CAREs:

$$\begin{aligned} &\Xi_{k,N}(i) \\ &= W_k + A^T\mathbf{L}^i\Big(\Xi_{k+1,N}(.)\Big)A - A^T\Big[\mathbf{L}^i\Big(\mathcal{N}(.)B^T\Xi_{k+1,N}(.)\Big)\Big]^T \\ &\times\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R_k + B^T\Xi_{k+1,N}(.)B\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\ &\times\mathbf{L}^i\Big(\mathcal{N}(.)B^T\Xi_{k+1,N}(.)\Big)A \end{aligned} \tag{19}$$

Remark 3. When ξ_k takes only two values: $0_{m\times m}$ or $1_{m\times m}$, then the CAREs (19) coincide with the CAREs given in [6] with $\Xi_{k,N}(0) = S_k$ and $\Xi_{k,N}(1) = R_k$.

Remark 4. For the case, when $\bar{v}_l = \bar{\mu}_l$, $\forall l \in \{0,1,...,(2^m-1)\}$, then the packet loss model becomes equivalent to the Bernoulli packet loss model and hence, CAREs (19) coincide with the CAREs given in Garone *et al.* [9].

In the rest of the paper, it is assumed that $W_k = W$ and $R_k = R$ for all k.

The following Lemma establishes the monotonicity of $\Xi_{k,N}(i,j)$; $\forall \in \{0,1,...,(2^m-1)\}$ and $\forall j \in \mathcal{D}$, which will be used in the following subsection.

Lemma 2. For $i \in \{0,1,...,(2^m-1)\}$ and $j \in \mathcal{D}$, $\Xi_{k,N}(i,j) \ge \Xi_{k+1,N}(i,j)$.

Proof: We prove the Lemma using induction. We have $\Xi_{N,N}(i,j) \ge 0 = \Xi_{N+1,N}(i,j)$ and let us assume $\Xi_{k+1,N}(i,j) \ge \Xi_{k+2,N}(i,j)$, $\forall i \in \{0,1,...,(2^m-1)\}$ and $j \in \mathcal{D}$, hence, from (11): $\mathbf{X}_{k+1,N}(i,j) \ge \mathbf{X}_{k+2,N}(i,j)$, $\forall i \in \{0,1,...,(2^m-1)\}$ and $j \in \mathcal{D}$.

Therefore, using (8) and (14)

$$\begin{aligned} &V_{k,N}\Big(x,\mathcal{N}(i),j\Big) = x^T\Xi_{k,N}(i,j)x \\ &= \min_u \Big[x^TWx + u^T\mathbf{L}^i\Big(\mathcal{N}(.)R\mathcal{N}(.)\Big)u \\ &+ \sum_{l=0}^{2^m-1}\mathcal{P}_l^i\Big\{\Big(A(j)x + B(j)\mathcal{N}(l)u\Big)^T\mathbf{X}_{k+1,N}(l,j) \\ &\times\Big(A(j)x + B(j)\mathcal{N}(l)u\Big)\Big\}\Big|\mathcal{I}_k\Big] \\ &\ge \min_u \Big[x^TWx + u^T\mathbf{L}^i\Big(\mathcal{N}(.)R\mathcal{N}(.)\Big)u \\ &+ \sum_{l=0}^{2^m-1}\mathcal{P}_l^i\Big\{\Big(A(j)x + B(j)\mathcal{N}(l)u\Big)^T\mathbf{X}_{k+2,N}(l,j) \\ &\times\Big(A(j)x + B(j)\mathcal{N}(l)u\Big)\Big\}\Big|\mathcal{I}_{k+1}\Big] \\ &= V_{k+1,N}\Big(x,\mathcal{N}(i),j\Big) = x^T\Xi_{k+1,N}(i,j)x \end{aligned} \tag{20}$$

So, $\Xi_{k,N}(i,j) \ge \Xi_{k+1,N}(i,j)$. □

Note 3. As $W_k = W$, $R_k = R$, $\forall k \in [0,N]$ and p_{ij} ($\forall i,j \in \mathcal{D}$) is time invariant, it is easy to see that $V_{k,N}\Big(x,\mathcal{N}(i),j\Big) = V_{0,N-k}\Big(x,\mathcal{N}(i),j\Big)$ for all $i \in \{0,1,...,(2^m-1)\}$, $j \in \mathcal{D}$, hence, $\Xi_{k,N}(i,j) = \Xi_{0,N-k}(i,j)$. Therefore, from *Lemma 2*, $\Xi_{0,N-k}(i,j) \ge \Xi_{0,N-k-1}(i,j)$. Since, k and N can be chosen arbitrarily, $\Xi_{0,L}(i,j) \ge \Xi_{0,L-1}(i,j)$ for $L \in [0,N]$.

B2. Infinite horizon control:

In this section, we obtain the infinite horizon optimal controller by considering $N \to \infty$.

Definition of stochastic stabilizability for classical jump linear systems is given in [13]. We extend this definition for jump systems with multiple lossy channels as follows.

Definition 1. System (1) is said to be stochastically stabilizable for some $(\bar{v}_l,\bar{\mu}_l)$, $l \in \{1,2,...,m\}$ if there exists a gain $\mathcal{K}(\bar{v}_l,\bar{\mu}_l)$ such that with control input $u_k = -\mathcal{K}(\bar{v}_l,\bar{\mu}_l)x_k$:

$$\sum_{k=0}^{\infty}\mathbb{E}\Big[||x_k||^2\Big] < \infty \tag{21}$$

By the following Lemma, convergence of the cost function (7) and hence, convergence of the $\Xi_{0,N}(i,j)$ as $N \to \infty$ will be established.

Lemma 3. Suppose system (1) is stochastically stabilizable for some $(\bar{v}_l,\bar{\mu}_l)$, then the solution to the infinite horizon problem is well defined.

Proof: Since, system (1) is stochastically stabilizable, there exists a control input $u_k = -\mathcal{K}(\bar{v}_l,\bar{\mu}_l)x_k$ such that $\sum_{k=0}^{\infty}\mathbb{E}\Big[||x_k||^2\Big] < \infty$. The infinite horizon cost incurred with this control input is given by:

$$\begin{aligned}
J_\infty &= \mathbb{E}\Big[\sum_{k=0}^{\infty}||x_k||_W^2 + ||\xi_k u_k||_R^2\Big] \\
&= \mathbb{E}\Big[\sum_{k=0}^{\infty}\Big(x_k^T W x_k \\
&+ x_k^T \mathcal{K}^T(\bar{v}_l,\bar{\mu}_l)\mathbf{L}^i\big(\mathcal{N}(.)R\mathcal{N}(.)\big)\mathcal{K}(\bar{v}_l,\bar{\mu}_l)x_k\Big)\Big] \\
&\leq \delta\mathbb{E}\Big[\sum_{k=0}^{\infty}||x_k||^2\Big] \\
&< \infty
\end{aligned} \tag{22}$$

where

$$\delta = \rho_{max}\Big\{W + \mathcal{K}^T(\bar{v}_l,\bar{\mu}_l)\mathbf{L}^i\Big(\mathcal{N}(.)R\mathcal{N}(.)\Big)\mathcal{K}(\bar{v}_l,\bar{\mu}_l)\Big\}$$

Clearly, we have: $J_\infty^* \leq J_\infty < \infty$.
Now, from (18), we can write the infinite horizon cost as follows:

$$J_\infty^* = \lim_{N\to\infty}\sum_{l=0}^{2^m-1}\hat{P}_l\Big[x_0^T\Xi_{0,N}(l,j)x_0\Big] < \infty \tag{23}$$

Hence, $\Xi_{0,N}(i,j)$, $\forall i \in \{0,1,...,(2^m-1)\}$, $\forall j \in \mathcal{D}$ is bounded as $N \to \infty$. Also, from *Note 3*, $\Xi_{0,N}(i,j)$ is a monotonically increasing function as N increases. Therefore, $\Xi_{0,N}(i,j) \to \bar{\Xi}(i,j)$ as $N \to \infty$, where $\bar{\Xi}(i,j)$ is the unique fixed point solution of the following CAREs.

$$\begin{aligned}
\bar{\Xi}(i,j) &= W + A^T(j)\mathbf{L}^i\Big(\bar{\mathbf{X}}(.,j)\Big)A(j) \\
&- A^T(j)\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(B^T(j)\bar{\mathbf{X}}(.,j)\Big)\Big]^T \\
&\times \Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R + B^T(j)\bar{\mathbf{X}}(.,j)B(j)\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\
&\times \mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\bar{\mathbf{X}}(.,j)\Big)A(j)
\end{aligned} \tag{24}$$

here, $\bar{\mathbf{X}}(i,j) = \sum_{t=1}^{\mathcal{M}}\Big\{p_{jt}\bar{\Xi}(i,t)\Big\}$ □

The following Lemma presents the infinite horizon version of *Lemma 1*.

Lemma 4. Suppose system (1) is stochastically stabilizable for some $(\bar{v}_l,\bar{\mu}_l)$, then:

(a) If $\xi_{k-1} = \mathcal{N}(i)$ and $r_k \in \mathcal{D}$, then the infinite horizon value function at any stage $k \in [0,\infty)$ can be expressed as follows:

$$V_{k,\infty}\Big(x_k,\mathcal{N}(i),j\Big) = x_k^T\bar{\Xi}(i,j)x_k \tag{25}$$

(b) The infinite horizon optimal control law is given by

$$\begin{aligned}
\bar{u}_k^* = &-\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R + B^T(j)\bar{\mathbf{X}}(.,j)B(j)\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\
&\times\mathbf{L}^i\Big(\mathcal{N}(.)B^T(j)\bar{\mathbf{X}}(.,j)\Big)A(j)x_k
\end{aligned} \tag{26}$$

(c) Optimal infinite horizon cost is given by:

$$J_\infty^* = \sum_{l=0}^{2^m-1}\hat{\mathcal{P}}_l\Big\{x_0^T\bar{\Xi}(l,r_0)x_0\Big\} \tag{27}$$

Proof: As $N \to \infty$, $\Xi_{k,N}(i,j)$ will no longer be function of k and $\Xi_{k,N}(i,j) \to \bar{\Xi}(i,j)$ for all $i \in \{0,1,...,(2^m-1)\}$ and $j \in \mathcal{D}$. So, replacing $\Xi_{k,N}(i,j)$ by $\bar{\Xi}(i,j)$, the Lemma can be proved by using the same argument used in *Lemma 1*. □

Remark 5. From *Lemma 4.*, the infinite horizon optimal controller for an LTI system can be expressed as follows:

$$\begin{aligned}
\bar{u}_k^* = &-\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R + B^T\bar{\Xi}(.)B\Big)\mathcal{N}(.)\Big)\Big]^{-1} \\
&\times\mathbf{L}^i\Big(\mathcal{N}(.)B^T\bar{\Xi}(.)\Big)Ax_k
\end{aligned} \tag{28}$$

where $\bar{\Xi}(i)$ for $0 \leq i \leq 2^m - 1$ is the fixed-point solution of the following CAREs:

$$\begin{aligned}
\bar{\Xi}(i) = W &+ A^T\mathbf{L}^i\Big(\bar{\Xi}(.)\Big)A - A^T\Big[\mathbf{L}^i\Big(\mathcal{N}(.)B^T\bar{\Xi}(.)\Big)\Big]^T \\
&\times\Big[\mathbf{L}^i\Big(\mathcal{N}(.)\Big(R + B^T\bar{\Xi}(.)B\Big)\mathcal{N}(.)\Big)\Big]^{-1}\mathbf{L}^i\Big(\mathcal{N}(.)B^T\bar{\Xi}(.)\Big)A
\end{aligned} \tag{29}$$

IV. Numerical Example

Let us consider the following MJLS:

$$x_{k+1} = A(r_k)x_k + B(r_k)u_k^a \tag{30}$$

where, $r_k = \{1,2\}$, $A(1) = 2.1$, $A(2) = 2.3$, $B(1) = B(2) = [1 \ \ 2]$, $W = 1$ and $R = I_{2\times 2}$. Let us take the switching probabilities as follows: $p_{11} = 0.4$, $p_{12} = 0.6$, $p_{22} = 0.75$ and $p_{21} = 0.35$. If the control packet arrival probabilities are chosen as $\bar{v}_1 = 0.75$, $\bar{v}_2 = 0.82$, $\bar{\mu}_1 = 0.85$, $\bar{\mu}_2 = 0.8$, then it can be observed that $\Xi_{0,N}(i,j)$ for $i \in \{0,1,2,3\}$ and $j \in \mathcal{D}$ converge, which is shown in Fig. 1. Therefore, the CAREs (24) admit unique fixed-point solution and hence, the infinite-horizon optimal cost converges. In Fig. 2., convergence of $\Xi_{0,N}(i,j)$ is demonstrated for the probabilities $\bar{v}_1 = 0.55$, $\bar{v}_2 = 0.65$, $\bar{\mu}_1 = 0.7$, $\bar{\mu}_2 = 0.73$. It can be observed that as the arrival probabilities are higher for the first case, $\Xi_{0,N}(i,j)$ converge faster for the first case (Fig. 1.) compared to the second case (Fig. 2.). If the arrival probabilities are further reduced to $\bar{v}_1 = 0.35$, $\bar{v}_2 = 0.45$, $\bar{\mu}_1 = 0.5$, $\bar{\mu}_2 = 0.4$, $\Xi_{0,N}(i,j)$ diverge, as $N \to \infty$ (Fig. 3.), hence, CAREs (24) do not admit fixed-point solution and infinite-horizon cost does not converge.

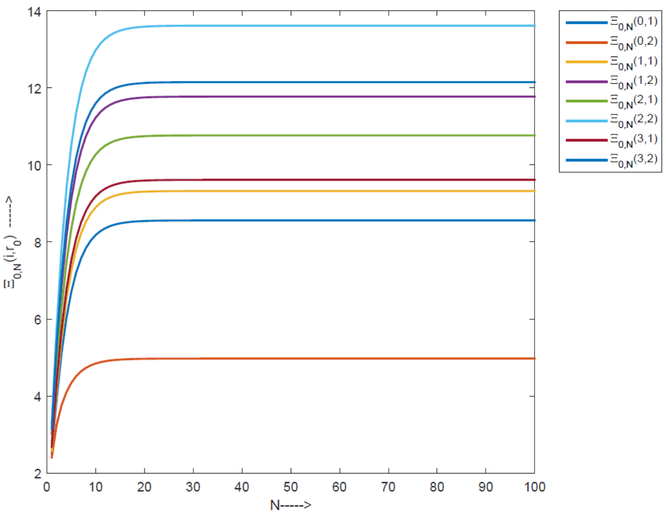

Fig. 2. Behavior of $\Xi_{0,N}(i,j)$ as horizon N increases with $\bar{v}_1 = 0.55$; $\bar{v}_2 = 0.65; \bar{\mu}_1 = 0.7; \bar{\mu}_2 = 0.73$

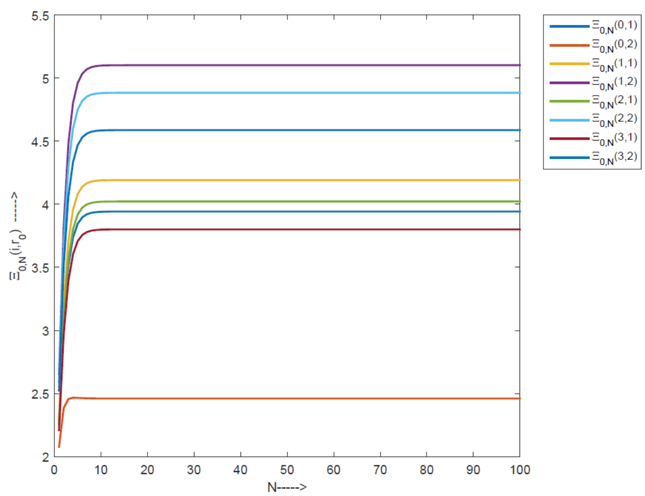

Fig. 1. Behavior of $\Xi_{0,N}(i,j)$ as horizon N increases with $\bar{v}_1 = 0.75$; $\bar{v}_2 = 0.82; \bar{\mu}_1 = 0.85; \bar{\mu}_2 = 0.8$

V. CONCLUSIONS

In this paper, we have investigated the linear quadratic optimal control of an MJLS over multiple channels considering correlated packet losses. The finite horizon and the infinite horizon controllers are designed considering the TCP-like case. Convergence of the infinite horizon cost function and hence, existence of the infinite horizon controller is also investigated. It is observed that if the control packet arrival probabilities are more than critical values, then the infinite horizon CAREs converge to the unique fixed-point solution. Moreover, as a special case, the finite horizon controller and the infinite horizon controller for an LTI system are also derived.

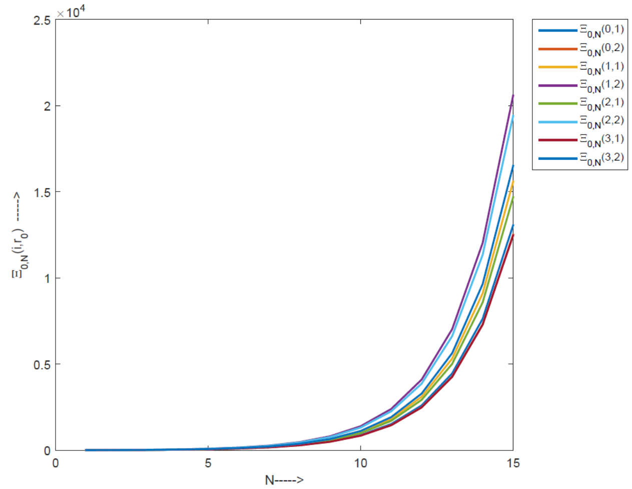

Fig. 3. Behavior of $\Xi_{0,N}(i,j)$ as horizon N increases with $\bar{v}_1 = 0.35$; $\bar{v}_2 = 0.45; \bar{\mu}_1 = 0.5; \bar{\mu}_2 = 0.4$

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