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Stability analysis and stabilization of the subclass of symmetric spatially interconnected systems*

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Abstract— **This paper develops new results on the stability and stabilization of symmetric active ladder circuits, which are a class of spatially interconnected systems. Ladder circuits can be considered as two–dimensional systems constituted as a homogenous (in the node structure and/or in parameters of nodes values) sequence of nodes (or cells) where information is propagated in two separate directions, i.e., along the time axis and along a space variable represented by a node number. In contrast to previous research in this general area, the independent energy source is placed in the center of the ladder. A new two-dimensional systems state-space model is developed for this case, leading to a new equivalent linear time-invariant model in one indeterminate obtained by applying a form of lifting along the nodes. Using this model description, the problems of stability testing and stabilization are addressed in a linear matrix inequality setting. A numerical case study is also given to illustrate the application of the new results.**

Keywords: Spatially interconnected systems, ladder circuits, stability and stabilization

I. INTRODUCTION

Ladder circuits can be considered as a particular case of spatially interconnected systems as are formed by a series of connected cells. Each cell is formed from resistances, reactances (impedances, in general) and also contains controlled and autonomous sources. The presence of controlled sources results in an active circuit and hence possibly unstable (without active sources a circuit is passive and hence always stable). Autonomous sources realize spatially distributed control inputs. In this paper, the circuit is assumed to have the same structure and parameter values that do not vary with the node number. The circuit to be analyzed is a series and/or parallel or mixed connection of these single cells.

Such ladder circuits can be a useful tool in, e.g., filter analysis and design, modeling delay lines, transmission lines, chains of transmission gates or long wire interconnections [1]. Another possible application is in deriving approximate models of, e.g., distributed parameter systems [2]. For a detailed discussion, refer on this general area see, e.g., [3] and references therein.

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In previous research, (see e.g. [4] and references therein) interconnected systems were considered, particularly non-symmetric ladder circuits represented as linear two dimensional (2D) hybrid (discrete–differential) systems for the case where the whole system is fed from the left hand side and loaded from the right hand side. The temporal dynamics were assumed to be governed by a continuous variable and the spatial dynamics (along the circuit nodes) by a discrete variable. This paper presents similar results for the symmetric case when the system is fed in the central circuit part and the possible loads are at the both left and right hand side ends.

For this new case, with the symmetry present, left and right hand side blocks can be combined into one common model, resulting in a bivariate, hybrid (discrete–differential) state-space model where the indeterminates are time (continuous variable) and node number (discrete variable). The symmetry, i.e. constant along the nodes, cell structure and their parameters, guarantees that the resulting overall model for the dynamics is node independent. Moreover, this model is characterized by non-causal spatial dynamics, i.e. the dynamics of a node dynamics depends on the dynamics of the elements it is composed from and those of the nodes on either side of it. Hence, lifting along the spatial variable is used to obtain a temporal, or 1D, model of the dynamics as the spatial dynamics are embedded in the overall structure. The same approach can also be applied to circuits with variable along the nodes cell structures and parameters - for the first results see e.g. [5]. Finally, stability conditions and stabilizing control laws are developed and verified by numerical example.

Throughout this paper $M \succ 0$ (respectively $\prec 0$) denotes a real symmetric positive (respectively negative) definite matrix. The null matrix with the required dimensions is denoted by 0 respectively. The symbol $\mathrm{diag}\{W_1, W_2, \cdots, W_M\}$ denotes a block diagonal matrix with diagonal blocks $W_1,\ W_2,\ \cdots, W_M$.

II. SYMMETRIC LADDER SYSTEMS AND THEIR STATE-SPACE MODELS

The symmetric systems considered are shown in Fig.1 where the right (left) hand side blocks are shown in Figs. 2 and 3, respectively.

In common with other areas of circuit analysis, the state variables are chosen as the inductor currents denoted by $i_L(p,t)$ (right-hand side) and $\bar{i}_L(p,t)$ (left-hand side), respectively, the capacitor voltages $U_C(p,t)$, or $\overline{U}_C(p,t)$, where p denotes the node number and $t \in \mathbf{R}$ is the time

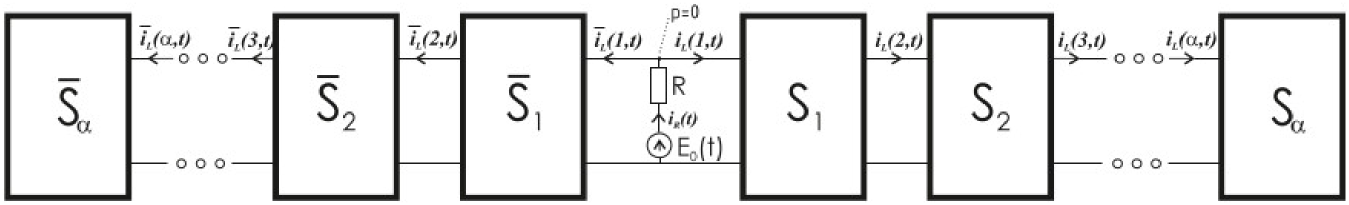

Fig. 1. Symmetric ladder system

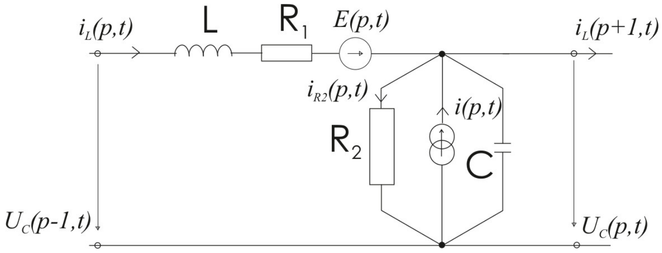

Fig. 2. Right hand side block

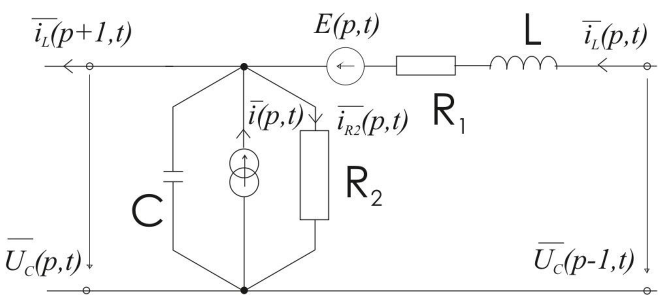

Fig. 3. Left hand side block

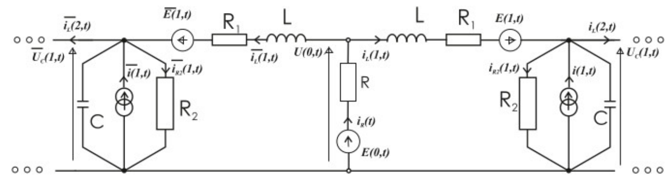

Fig. 4. Central block

variable. A variable on the left hand side of the circuit is denoted by $\overline{\omega}$ and bω denotes a right hand side variable. Also the elements in the circuit are taken as constants. The circuit is also symmetric and contains the same number, α, of left and right hand side nodes, i.e., $p = 1, 2, \cdots, \alpha$ and the central node is denoted by $p = 0$ (see Fig. 4).

Introduce the right and left hand side state vectors as

$$x_R(p,t) = \begin{bmatrix} U_C(p,t) \\ i_L(p,t) \end{bmatrix}, \;\; x_L(p,t) = \begin{bmatrix} \overline{U}_C(p,t) \\ \bar{i}_L(p,t) \end{bmatrix}, \qquad (1)$$

respectively. Moreover, the circuits considered, denoted $RLCM$, with no controlled sources cannot be unstable. Assume, therefore that current sources $i(p,t)$ and $\bar{i}(p,t)$ are controlled abusing

$$i(p,t) = \gamma U_c(p+1,t), \;\; \bar{i}(p,t) = \gamma \overline{U}_c(p+1,t), \qquad (2)$$

where γ is a known real number and hence the resulting active system can be unstable for some values of this number. The voltage sources $E(p,t)$ ($\overline{E}(p,t)$ for the left hand side of the circuit) are independent and serve as the distributed control for the circuit, termed a system from this point onwards.

A. Central part of the system

Consider first the central part of the system, shown in Fig. 4. To obtain the governing state-space equations first apply Kirchhoff's voltage law to the right (left) hand side branch between nodes 0 and 1, respectively, to obtain

$$\begin{aligned} U(0,t) - U_C(1,t) &= L\frac{d}{dt}i_L(1,t) + R_1 i_L(1,t) - E(1,t), \\ U(0,t) - \overline{U}_C(1,t) &= L\frac{d}{dt}\bar{i}_L(1,t) + R_1 \bar{i}_L(1,t) - \overline{E}(1,t). \end{aligned}$$

Next, applying Kirchhoff's current law to node 1 from the right and left hand sides, respectively, gives

$$\begin{aligned} C\frac{d}{dt}U_C(1,t) &= i(1,t) - \frac{1}{R_2}U_C(1,t) + i_L(1,t) - i_L(2,t), \\ C\frac{d}{dt}\overline{U}_C(1,t) &= \bar{i}(1,t) - \frac{1}{R_2}\overline{U}_C(1,t) + \bar{i}_L(1,t) - \bar{i}_L(2,t) \end{aligned}$$

where

$$U(0,t) = E(0,t) - R(i_L(1,t) + \bar{i}_l(1,t). \qquad (3)$$

Combining these equations, introducing (2) and applying (1) yields the following state-space equation for the right hand side block S_1

$$\begin{aligned} \frac{d}{dt}x_R(1,t) = &\begin{bmatrix} -\frac{1}{R_2 C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1+R_2}{L} \end{bmatrix} x_R(1,t) \\ &+ \begin{bmatrix} \frac{\gamma}{C} & -\frac{1}{C} \\ 0 & 0 \end{bmatrix} x_R(2,t) + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{R_1}{L} \end{bmatrix} x_L(1,t) \\ &+ \begin{bmatrix} 0 \\ \frac{1}{L}(E(0,t)+E(1,t)) \end{bmatrix} \qquad (4) \\ = &\mathcal{A}_{02}x_R(1,t) + \mathcal{A}_{03}x_R(2,t) + \mathcal{A}_{01}x_L(1,t) \\ &+ \mathcal{B}u_R(1,t), \end{aligned}$$

where $\mathcal{B} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$. The corresponding model for the left hand side block $\overline{S}_1$ is

$$\begin{aligned}\frac{d}{dt}x_L(1,t) = &\begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1+R_2}{L} \end{bmatrix} x_L(1,t) \\ &+ \begin{bmatrix} \frac{\gamma}{C} & -\frac{1}{C} \\ 0 & 0 \end{bmatrix} x_L(2,t) + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{R_1}{L} \end{bmatrix} x_R(1,t) \\ &+ \begin{bmatrix} 0 \\ \frac{1}{L}(E(0,t)+\overline{E}(1,t)) \end{bmatrix} \\ =&\mathcal{A}_{02}x_L(1,t) + \mathcal{A}_{03}x_L(2,t) + \mathcal{A}_{01}x_R(1,t) \\ &+ \mathcal{B}u_L(1,t),\end{aligned} \tag{5}$$

with $\mathcal{B}$ as in the previous model. Hence, the input signal to the blocks S_1 and $\overline{S}_1$, respectively, is given as

$$u_R(1,t) = E(0,t) + E(1,t) \text{ and } u_L(1,t) = E(0,t) + \overline{E}(1,t). \tag{6}$$

Both these sets of equations have an identical form, except that signals denoted by v are replaced by $\overline{v}$ and vice versa (except for node 1).

B. Right and left hand side blocks

The right hand side structure is shown in Fig. 5 and the left hand structure in Fig. 6. Applying a similar approach as in previous subsection to nodes $p = 2, 3, \ldots, \alpha$ of the right and side blocks S_p, gives, as in previous work, (e.g., [3] and references therein) the following 2D differential-discrete state-space model

$$\begin{aligned}\frac{d}{dt}x_R(p,t) = &\mathcal{A}_1 x_R(p-1,t) + \mathcal{A}_2 x_R(p,t) \\ &+\mathcal{A}_3 x_R(p+1,t) + \mathcal{B}u_R(p,t),\end{aligned} \tag{7}$$

where $u_R(p,t) = E(p,t)$, denote the distributed along nodes input vectors. The matrices in the model (7) are

$$\mathcal{A}_1 = \begin{bmatrix} \frac{\gamma}{C} & 0 \\ \frac{1}{L} & 0 \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}, \mathcal{A}_3 = \begin{bmatrix} 0 & -\frac{1}{C} \\ 0 & 0 \end{bmatrix} \tag{8}$$

and the matrix $\mathcal{B}$ is again given by (4).

By a similar analysis, the 2D differential-discrete state-space model is for the left hand side blocks ($\overline{S}_p$) is

$$\begin{aligned}\frac{d}{dt}x_L(p,t) =&\mathcal{A}_1 x_L(p-1,t) + \mathcal{A}_2 x_L(p,t) \\ &+ \mathcal{A}_3 x_L(p+1,t) + \mathcal{B}u_L(p,t),\end{aligned} \tag{9}$$

where $u_L(p,t) = \overline{E}(p,t)$ and model matrices are again given by (8). To complete the description, the following boundary conditions are assumed

$$x_R(p,0) = 0, \;\; x_L(p,0) = 0, \;\;\; 1 \le p \le \alpha. \tag{10}$$

III. The 1D Equivalent Model

First, the structure introduced by the symmetrically placed cells models is exploited by introducing the 'common' state and input vectors together with the augmented input vector as

$$x(p,t) = \begin{bmatrix} x_R(p,t) \\ x_L(p,t) \end{bmatrix}, \; u(p,t) = \begin{bmatrix} u_R(p,t) \\ u_L(p,t) \end{bmatrix},$$
$$\tilde{u}(p,t) = \begin{bmatrix} u(p,t) \\ u(p,t) \end{bmatrix}$$

Then using (4)-(5)

$$\begin{aligned}\frac{d}{dt}x(1,t) = &\begin{bmatrix} \mathcal{A}_{02} & \mathcal{A}_{01} \\ \mathcal{A}_{01} & \mathcal{A}_{02} \end{bmatrix} x(1,t) + \begin{bmatrix} \mathcal{A}_{03} & 0 \\ 0 & \mathcal{A}_{03} \end{bmatrix} x(2,t) \\ &+ \begin{bmatrix} \mathcal{B} \\ \mathcal{B} \end{bmatrix} u(1,t)\end{aligned}$$

or

$$\begin{aligned}\frac{d}{dt}x(1,t) = &\begin{bmatrix} \mathcal{A}_{02} & \mathcal{A}_{01} \\ \mathcal{A}_{01} & \mathcal{A}_{02} \end{bmatrix} x(1,t) + \begin{bmatrix} \mathcal{A}_{03} & 0 \\ 0 & \mathcal{A}_{03} \end{bmatrix} x(2,t) \\ &+ \begin{bmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{B} \end{bmatrix} \tilde{u}(1,t) \\ =&\widehat{\mathcal{A}}_2 x(1,t) + \widehat{\mathcal{A}}_3 x(2,t) + \widetilde{\mathcal{B}}\tilde{u}(1,t).\end{aligned} \tag{11}$$

Similarly, for (7)-(9)

$$\begin{aligned}\frac{d}{dt}x(p,t) = &\begin{bmatrix} \mathcal{A}_1 & 0 \\ 0 & \mathcal{A}_1 \end{bmatrix} x(p-1,t) + \begin{bmatrix} \mathcal{A}_2 & 0 \\ 0 & \mathcal{A}_2 \end{bmatrix} x(p,t) \\ &+ \begin{bmatrix} \mathcal{A}_3 & 0 \\ 0 & \mathcal{A}_3 \end{bmatrix} x(p+1,t) + \begin{bmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{B} \end{bmatrix} \tilde{u}(p,t) \\ =&\widetilde{\mathcal{A}}_1 x(p-1,t) + \widetilde{\mathcal{A}}_2 x(p,t) + \widetilde{\mathcal{A}}_3 x(p+1,t) \\ &+ \widetilde{\mathcal{B}}\tilde{u}(p,t).\end{aligned}$$

This approach can also be used for circuits constructed from different in structure and/or parameter values cell if the number of left and right hand side cells is the same. In such a case, however, the overall system is not node number constant. If the number of left and right hand side cells are not the same, a number of models will remain single as there is no symmetric counterpart.

The next step is to apply the lifting along the spatial variables, which produces the possibly high dimensional 1D system description, for which, due to the spatial symmetry of the system, one description for the left and right hand side cells models can be used. The lifted model is defined in terms of the following state and input super-vectors $\mathbf{x}(t)$ and $\mathbf{u}(t)$, i.e.:

$$\begin{aligned}\mathbf{x}(t) \;&= \left[x(1,t)^T, \; x(2,t)^T, \; \ldots x(\alpha,t)^T\right]^T, \\ \mathbf{u}(t) \;&= \left[\tilde{u}(1,t)^T, \; \tilde{u}(2,t)^T, \ldots \tilde{u}(\alpha,t)^T\right]^T.\end{aligned} \tag{12}$$

Hence an augmented state-model for the system dynamics is

$$\frac{d}{dt}\mathbf{x}(t) = \Phi\mathbf{x}(t) + \Psi\mathbf{u}(t), \tag{13}$$

where

$$\Phi = \begin{bmatrix} \widehat{\mathcal{A}}_2 & \widehat{\mathcal{A}}_3 & 0 & & & 0 \\ \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{A}}_2 & \widetilde{\mathcal{A}}_3 & 0 & & \\ 0 & \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{A}}_2 & \widetilde{\mathcal{A}}_3 & 0 & \\ & & \ddots & \ddots & \ddots & 0 \\ & & 0 & \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{A}}_2 & \widetilde{\mathcal{A}}_3 \\ 0 & & & 0 & \widetilde{\mathcal{A}}_1 & \widetilde{\mathcal{A}}_2 \end{bmatrix}, \tag{14}$$
$$\Psi = \text{diag}\{\widetilde{\mathcal{B}}, \widetilde{\mathcal{B}}, \cdots, \widetilde{\mathcal{B}}\}$$

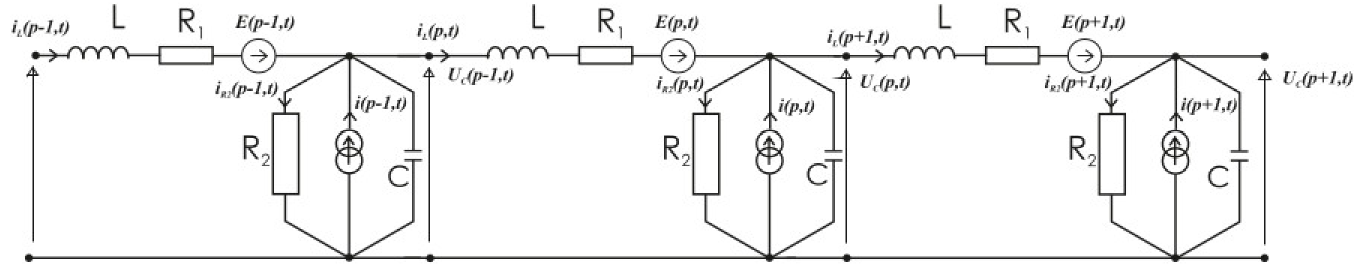

Fig. 5. Right hand structure

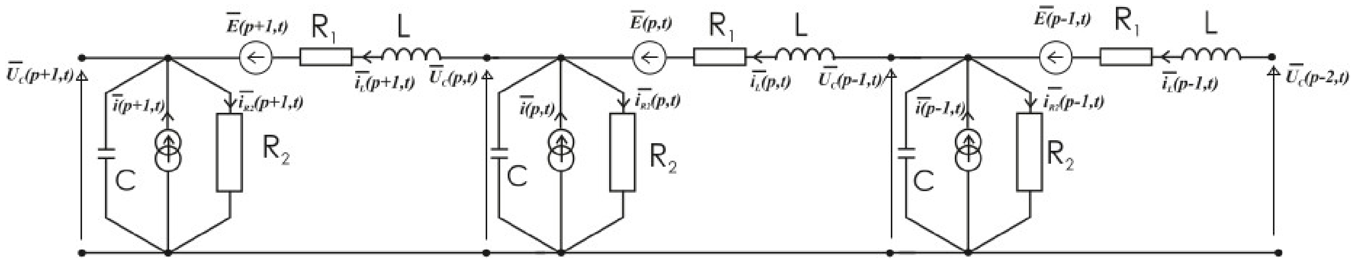

Fig. 6. Left hand structure

and also Φ is a block Toeplitz matrix. In this representation the dynamics in p (along the nodes) are absorbed into the super-vectors.

IV. Stability analysis and stabilization

The following result characterizes the stability properties of the systems considered.

Lemma 1: [6] The system described by (13) with no control inputs is stable if and only if $\exists\, \mathcal{P} \succ 0$, such that the following Linear Matrix Inequality (LMI) condition holds

$$\Phi^T \mathcal{P} + \mathcal{P}\Phi \prec 0. \tag{15}$$

The assumption of a fully populated matrix $\mathcal{P}$ in Lemma 1 can lead to computational problems when the length of the circuit α is large. However, the 1D model matrix Φ has a block tridiagonal structure and hence it is not necessary to assume that $\mathcal{P}$ is fully populated. This leads to the following theorem, which gives a sufficient condition for stability, where the dimensions of the matrices involved have been much reduced.

Theorem 1: The system described by (13) with no control inputs is stable, if $\exists\, \mathcal{P} = \mathrm{diag}(P_1, P_2, \cdots, P_\alpha) \succ 0$, $P_p = \mathrm{diag}(P_p^*, P_p^*)$, $p = 1, 2, \ldots, \alpha$, such that the following LMI condition holds

$$\Phi^T \mathcal{P} + \mathcal{P}\Phi \prec 0. \tag{16}$$

Remark 1: Given the structure of the system matrix Φ it follows that, except for the first tri-diagonal blocks (which relate to node no. 1), every block row is repeated but shifted one block column to the right relative to its predecessor. Hence, further reduction of the dimensions of the matrices (the total number of decision variables) is still possible, e.g., $P_p = \mathrm{diag}(P, P)$, $p = 2, 3, \ldots, \alpha$. However, such simplification may come at the cost of increasing the conservativeness of the LMIs.

A. Controller design

Consider the following variable with the node numbers control law

$$\begin{aligned} u(1,t) =& u^2(1,t) + u^3(3,t)\\ =& K^2(1)x(1,t) + K_3(2)x(2,t)\\ u(p,t) =& u^1(p,t) + u^2(p,t) + u^3(p,t)\\ =& K^1(p)x(p-1,t) + K^2(p)x(p,t)\\ &+ K^3(p)x(p+1,t) \end{aligned} \tag{17}$$

or in equivalent 1D model form

$$\mathbf{u}(t) = \mathcal{K}\mathbf{x}(t), \tag{18}$$

where

$$\mathcal{K} = \begin{bmatrix} K^2(1) & K^3(1) & 0 & \ldots & \ldots & 0 \\ K^1(2) & K^2(2) & K^3(2) & 0 & \ddots & \vdots \\ 0 & K^1(3) & K^2(3) & K^3(3) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & K^1(\alpha-1) & K^2(\alpha-1) & K^3(\alpha-1) \\ 0 & \ldots & \ldots & 0 & K^1(\alpha) & K^2(\alpha) \end{bmatrix} \tag{19}$$

and

$$K^2(1) = \begin{bmatrix} K_2^2(1) & K_3^2(1) \\ K_3^2(1) & K_2^2(1) \end{bmatrix},$$
$$K^2(p) = \begin{bmatrix} K_2^2(p) & 0 \\ 0 & K_2^2(p) \end{bmatrix},\ p = 2,\ 3,\ \ldots,\ \alpha$$
$$K^1(p) = \begin{bmatrix} K_1^1(p) & 0 \\ 0 & K_1^1(p) \end{bmatrix},\ p = 1,\ 2,\ \ldots,\ \alpha$$
$$K^3(p) = \begin{bmatrix} K_3^3(p) & 0 \\ 0 & K_3^3(p) \end{bmatrix},\ p = 1,\ 2,\ \ldots,\ \alpha.$$

Theorem 2: Suppose that the control law (18) is applied to a system described by (13). Then, under the standard controllability assumption, the resulting controlled model is stable, if $\exists\ \mathcal{Q} = \operatorname{diag}(Q_1, Q_2, \ldots, Q_\alpha) \succ 0$, where $Q(p) = \operatorname{diag}(Q_p^*, Q_p^*),\ p = 1,\ 2,\ \ldots,\ \alpha$ and

$$\mathcal{N} = \begin{bmatrix} N^2(1) & N^3(1) & 0 & \ldots & \ldots & 0 \\ N^1(2) & N^2(2) & N^3(2) & 0 & \ddots & \vdots \\ 0 & N^1(3) & N^2(3) & N^3(3) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & N^1(\alpha-1) & N^2(\alpha-1) & N^3(\alpha-1) \\ 0 & \ldots & \ldots & 0 & N^1(\alpha) & N^2(\alpha) \end{bmatrix} \tag{20}$$

and

$$N^2(1) = \begin{bmatrix} N_2^2(1) & N_3^2(1) \\ N_3^2(1) & N_2^2(1) \end{bmatrix},$$
$$N^2(p) = \begin{bmatrix} N_2^2(p) & 0 \\ 0 & N_2^2(p) \end{bmatrix},\ p = 2,\ 3,\ \ldots,\ \alpha$$
$$N^1(p) = \begin{bmatrix} N_1^1(p) & 0 \\ 0 & N_1^1(p) \end{bmatrix},\ p = 1,\ 2,\ \ldots,\ \alpha$$
$$N^3(p) = \begin{bmatrix} N_3^3(p) & 0 \\ 0 & N_3^3(p) \end{bmatrix},\ p = 1,\ 2,\ \ldots,\ \alpha$$

of compatible dimensions, such that the following LMI condition holds

$$\mathcal{Q}\Phi^T + \Phi\mathcal{Q} + \mathcal{N}^T\Psi + \Psi\mathcal{N} \prec 0. \tag{21}$$

If the LMI of (21) is feasible, the control matrix is given by

$$\mathcal{K} = \mathcal{N}\mathcal{Q}^{-1}. \tag{22}$$

Proof. Applying the condition of (16) to the system model with the control law (20) applied gives

$$(\Phi + \Psi\mathcal{K})^T\mathcal{P} + \mathcal{P}(\Phi + \Psi\mathcal{K}) \prec 0.$$

Left and right-multiplying this last condition by $\mathcal{Q} = \mathcal{P}^{-1}$ gives

$$\mathcal{Q}\Phi^T + \mathcal{Q}\mathcal{K}^T\Psi^T + \Phi\mathcal{Q} + \Psi\mathcal{K}\mathcal{Q} \prec 0.$$

Finally, setting $\mathcal{N} = \mathcal{K}\mathcal{Q}$ gives (21). □

Remark 2: In this last result, the controller matrices for any node can, in general, vary. This is justified especially for $K^2(1)$, i.e. for the central nodes $p = 1$, but for the following nodes controller matrices can be assumed to be the same for the rest of the system, i.e. $K^1(p) = K^1,\ K^3(p) = K^3,\ p = 1, 2, \ldots, \alpha,\ K^2(p) = K^2,\ p = 2, 3, \ldots, \alpha$. This could introduce some extra conservativeness of the underlying LMI, but the dimensions of the matrices involved would be smaller.

V. Numerical example

Consider the active circuit built from elements with the following values $C = 5 \times 10^{-4}\ [F]$, $L = 1 \times 10^{-2}\ [H]$, $R_1 = 100\ [\Omega]$, $R_2 = 200\ [\Omega]$, $R = 50\ [\Omega]$ and the controlled current sources with $\gamma = 0.001$, see (2). The length of the each symmetric parts of ladder is set to $\alpha = 20$. This model is unstable since the real values of some eigenvalues of Φ are positive and no solution for the LMI of Theorem 1 exists. Note however that the instability is due to the presence of the controlled sources as, if these are absent, then system is passive and hence stable for all remaining parameter values.

The instability of the model can also be seen in simulation results. Moreover, since all boundary conditions are assumed to be zero, free evolution of the system is of no interest. Instead, assume that during simulation a small input signal $E(0,t) = 1\,[V]$ appears at $t = 0.1\ [s]$ and lasts until $t = 0.2\ [s]$. The evolution of the system in this case is shown in Figure 7. It can be seen that the voltages across the capacitors increase with time, long after the input signal has returned to zero.

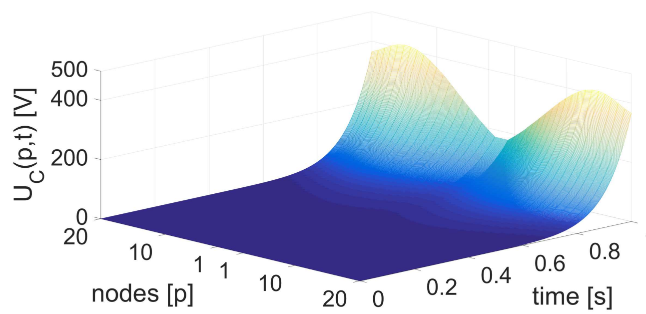

Fig. 7. Simulation results for the uncontrolled active circuit — $U_C(p,t)$ over time and nodes

Application of Theorem 2 provides stabilizing controllers. As an example, consider the controller matrix based on Remark 2. In this case, solving the LMI (22) gives

$$Q_p^* = 10^{-8} \times \begin{bmatrix} 0.0060 & -0.0047 \\ -0.0047 & 0.1947 \end{bmatrix},$$
$$N_2^2(p) = 10^{-5} \times \begin{bmatrix} 0.0013 & -0.1566 \end{bmatrix},$$
$$N_3^2(1) = 10^{-5} \times \begin{bmatrix} -0.0027 & 0.1565 \end{bmatrix},$$
$$N_3^3(p) = 10^{-5} \times \begin{bmatrix} -0.0026 & 0.1566 \end{bmatrix},$$
$$N_1^1(p) = 10^{-8} \times \begin{bmatrix} 0.9060 & 0.0344 \end{bmatrix},$$
$$p = 1,\ 2,\ \ldots,\ \alpha.$$

Hence the controller matrices are

$$K_2^2(p) = \begin{bmatrix}-423 & -815\end{bmatrix}, \; K_2^3(1) = \begin{bmatrix}184 & 808\end{bmatrix},$$
$$K_3^3(p) = \begin{bmatrix}206 & 809\end{bmatrix}, \; K_1^1(p) = \begin{bmatrix}154 & 4\end{bmatrix},$$
$$p = 1, 2, \ldots, \alpha.$$

To illustrate the behavior of the controlled circuit suppose that, as in the previous case, an input signal $E(0,t) = 1\,[V],\ t \in< 0.1,\, 0.2 >\ [s]$ is applied. Figures 8 and 9 give the resulting dynamics of state variables for the controlled circuit.

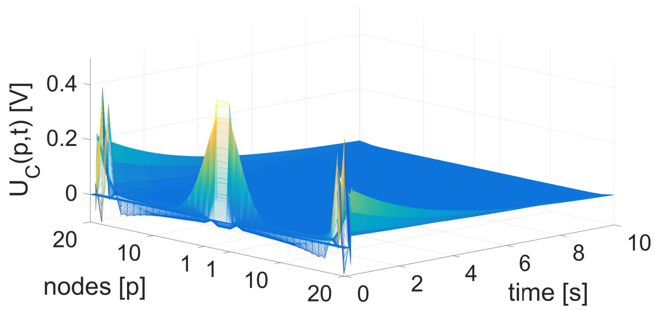

Fig. 8. Simulation results for the controlled circuit — $U_C(p,t)$ over time and nodes

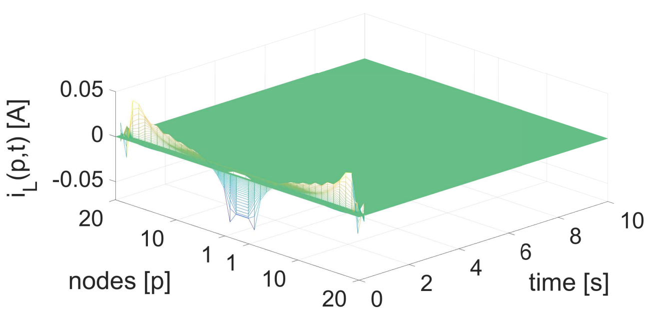

Fig. 9. Simulation results for the controlled circuit — $i_L(p,t)$ over time and nodes

Areas for possible future research include robust control, time discretization and iterative learning control design. Effort could also be applied to extending these results for ladder circuits to multi-mass systems assembled as a long chain of

The control action applied clearly results in stable behaviour.

VI. Conclusions

This paper has developed a 1D state-space model for a specific class of spatially interconnected system, i.e., a symmetrical active circuit. Based on Kirchhoff's laws, the 2D dynamic model has been derived with one temporal and one spatial (the node number) indeterminates. Next, taking advantage of the system symmetry and applying the lifting along the nodes approach, the equivalent 1D model has been produced. This has led on to the development of stability analysis/tests and stabilizing control law design, where the required computations are LMI based.
subsystems, which have numerous engineering applications, see, e.g., [7].

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