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Study on Stability of General Discrete-Time Stochastic Difference Systems*

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***Abstract*— This electronic document is a live template. The various components of your paper [title, text, heads, etc.] are already defined on the style sheet, as illustrated by the portions given in this document.**

I. Extended Abstract

Since Lyapunov initiated his direct method or called Lyapunov's second method about system stability in 1892, Lyapunov's stability theory has been occupying a dominant position in mathematics and modern control theory, this is due to the following twofold reasons: firstly, stability is the first of all to be considered in system analysis and design. Secondly, Lyapunov's second method has been the most powerful tool in testing a general nonlinear system to be stable or unstable, which avoids solving ordinary differential equations analytically. As applied in deterministic ordinary differential equations and difference equations [1], Lyapunov' second method has been successfully extended to study the stability of stochastic differential systems; see [2], [3], [4]. However, there seems no systematic monograph or work on stability of general discrete-time stochastic difference equations corresponding to [3], [4].

In this paper, we will report our recent studies in stability of general discrete-time stochastic systems. At present, difference systems become more and more important, this is because many engineering problems can be modeled by stochastic difference equations [5]. In addition, along with the development of computer technology, the numerical solution for stochastic differential equations has become a popular research topic; see, e.g. [6], which is in fact a discretized process. In recent years, the study on discrete stochastic systems has attracted many scholars' attention [7], [8], [9]. In [10], we presented a discrete stochastic maximal principle for the following optimal control problem: minimize the cost functional

$$J(u,x_0)=\sum_{k=0}^{N-1}\mathcal{E}l(x_k,u_k)+\mathcal{E}h(x_N) \tag{1}$$

subject to

$$x_{k+1}=g(x_k,u_k)+\sigma(x_k,u_k)\omega_k,\quad x_0\in\mathcal{R}^n, k=1,2,\cdots,N-1, \tag{2}$$

which is a counterpart to that of continuous-time Itô systems [11]. In [12], we extended the classical LaSalle invariance principle [13] to the following discrete-time time-varying stochastic system:

$$\begin{cases} x_{k+1}=F_k(x_k,\omega_k),\\ x_0\in\mathcal{R}^n, k\in\mathcal{N}:=\{0,1,2,\cdots\}\end{cases} \tag{3}$$

which can also be viewed as a discrete version of [14]. As applications, the nonlinear optimal regulator problem is also solved in [12] for the optimized cost functional

$$\min_{u\in\mathcal{U}}\{J(u,x_0):=\sum_{k=0}^{\infty}\mathcal{E}[l_k(x_k,u_k)]\}. \tag{4}$$

under the constraint of (3). Except for [12], we refer the reader to [15], [16], [17] for optimal control of some special kinds of discrete stochastic systems such as

$$x_{k+1}=f(x_k)+g(x_k)u_k+h(x_k)\omega_k.$$

The aim of this paper is to give several stability criteria for the following system

$$\begin{cases} x_{k+1}=F_k(x_k,\omega_k), F_k(0,y)\equiv 0,\forall y\in\mathcal{R}^d,\\ x_s=x\in\mathcal{R}^n, k\in\mathcal{N}_s:=\{s,s+1,s+2,\cdots\}\end{cases} \tag{5}$$

where in (5), x_s ($s\in\mathcal{N}$) is the initial state, $\{x_k\}_{k\in\mathcal{N}_s}$ is the $\mathcal{R}^n$-valued state variable sequence. $\{\omega_k\}_{k\in\mathcal{N}_s}$ is an independent $\mathcal{R}^d$-valued random variable sequence defined on a given complete probability space $(\Omega,\mathcal{F},\mathcal{P})$. $F_k:\mathcal{R}^n\times\mathcal{R}^d\mapsto\mathcal{R}^n$ is a continuous function for $k\in\mathcal{N}_s$. For system (5), we denote its solution sequence as $\{x_k^{s,x}\}_{k\in\mathcal{N}_s}$. We first introduce some stability definitions such as stability in probability, stochastic asymptotic stability, stochastic asymptotic stability in the large. Concretely speaking, the trivial solution $x_k\equiv 0$ of system (5) is said to be stable in probability if for any $s\in\mathcal{N}$ and $r>0$, we have

$$\lim_{\|x\|\to 0}\mathcal{P}\{\sup_{k\in\mathcal{N}_s}\|x_k^{s,x}\|>r\}=0. \tag{6}$$

If (6) holds uniformly in $s\in\mathcal{N}$, then the trivial solution $x_k\equiv 0$ of (5) is said to be uniformly stable in probability. The trivial solution $x_k\equiv 0$ of system (5) is said to be stochastically asymptotically stable in probability if it is stochastically stable and moreover

$$\lim_{\|x\|\to 0}\mathcal{P}\{\lim_{k\to\infty}x_k^{s,x}=0\}=1. \tag{7}$$

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If (7) holds for all $(s,x) \in \mathcal{N} \times \mathcal{R}^n$, i.e., if (7) is replaced by

$$\mathcal{P}\{\lim_{k\to\infty} x_k^{s,x} = 0\} = 1, \quad (8)$$

then the trivial solution $x_k \equiv 0$ of system (5) is called stochastically asymptotically stable in the large. All the above introduced concepts are consistent with those of continuous-time Itô systems in their formulations [4], [2], however, it is more difficult to give practical criteria to test the above stabilities. The reference [18] tried to discuss the stability of system (5) by following the line of [2], where $LV(x,t) \leq 0 (< 0)$ in [2] is replaced by

$$\mathcal{E}\Delta V(x_k^{s,x}) = \mathcal{E}V(x_{k+1}^{s,x}) - \mathcal{E}V(x_k^{s,x}) \leq 0 (< 0). \quad (9)$$

It should be pointed out that condition (9) is not easily verified because it contains the mathematical expectation of the state variable $x_k^{s,x}$, while similar conditions has been used in many references; see, e.g. [19], [20], [21]. Moreover, there is something wrong in the proofs of the main theorems of [18].

In order to improve the previous results and obtain efficient criteria for discrete stochastic stability, new techniques should be introduced. In this paper, by using Doob's martingale theory and stopping time theorem for supermartingale, we have successfully generalized some classical stability theorems existing in Itô systems to general discrete stochastic system (5), which are expected to be useful in stochastic H_∞ control and network control. Concretely speaking, the main contributions of this paper is as follows:

- Give a sufficient condition for the system (5) to be stable in probability. That is, if there exists a positive definite Lyapunov sequence $\{V_k(x) := V(k,x)\}_{k\in\mathcal{N}_s}$, $V_k(x) \in \mathcal{N}_s \times S_r$, $s \in \mathcal{N}$, $S_r := \{x : \|x\| < r\}$, such that
$$\Delta V_k(x) = \mathcal{E}V_{k+1}(F(x,\omega_{k+1})) - V_k(x) \leq 0, \quad (10)$$
then system (5) is stable in probability.
- Moreover, if $\{V_k(x)\}_{k\in\mathcal{N}}$ has an infinitesimal upper limit, i.e.,
$$\lim_{\|x\|\to 0} \sup_{k\in\mathcal{N}} V_k(x) = 0,$$
then the trivial solution of system (5) is uniformly stable in probability.
- If there exists a positive definite decrescent Lyapunov sequence $\{V_k(x)\}_{k\in\mathcal{N}_s}$ on $\mathcal{N}_s \times S_r$, such that $\Delta V_k(x)$ is negative definite in Lyapunov's sense, then the trivial solution of system (5) is stochastically asymptotically stable in probability.
- If there exists a positive definite decrescent radially unbounded Lyapunov sequence $\{V_k(x)\}_{k\in\mathcal{N}_s}$ on $\mathcal{N}_s \times \mathcal{R}^n$, such that $\Delta V_k(x)$ is negative definite in Lyapunov's sense, then the trivial solution of system (5) is stochastically asymptotically stable in the large.

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