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Distributed Recursive Filter for Systems with Multiplicative Noises

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Abstract— **This paper studies the distributed state estimation problem for a class of linear time-invariant (LTI) systems with multiplicative noises over multi-agent networks. First, adaptive fusion matrices with distributed strategy is considered in terms of consistency, which provides an upper bound for mean square error matrix of each agent and the bound is minimized by the designed time-varying filter gain. Based on the designed fusion matrices and filter gain, a distributed robust Kalman filter is proposed. Moreover, under mild conditions of collective observability and network topology, it is shown that the mean square error matrices of agents are uniformly upper bounded and the information matrix of the network are uniformly lower bounded by constant matrices both through finite time, which is revealed to be the summation of collective observability parameter and network scale. Finally, the asymptotic unbiasedness of the estimation of each agent is shown. Numerical simulation shows the effectiveness of the proposed algorithm.**

I. Introduction

State estimation problems or filtering problems have been studied for several decades due to their close relationship with parameter identification, signal reconstruction, target monitoring and control design [1]. In recent years, the research of distributed state estimation problems are drawing more and more attention due to the various applications in engineering systems such as communication networks, sensor networks and smart grids [2], [3]. Due to the complexity of environment and the uncertainties of systems, we have to deal with some certain situations [4], [5], [6], which prevent the real-time estimation for unknown stochastic dynamics.

Multiplicative noise, as one of the common system uncertainties, exists in many cases like signal transmission and sampling, amplitude modulation, etc. In centralized frameworks, [6] studied the multiplicative noise for a kind of linear discrete systems and provided a recursive state estimator. In [7], the authors studied a class of discrete time-varying systems with multiplicative noises and norm-bounded uncertainties. Sufficient conditions to guarantee an optimized upper bound of estimation covariance were established in terms of two Riccati difference equations. [8] proposed a recursive state estimator for a linear time-varying system with parametric uncertainties and stochastic measurement droppings, where theoretical analysis on the convergence of the proposed robust estimator was given. Meanwhile, in the decentralized frameworks, some networked robust filters have been proposed. The authors of [9] studied the distributed fusion problem for uncertain systems with correlated noises and they provided a weighted robust Kalman filter involving a recursive computation of cross-covariance matrix between any two sensors. [10] discussed the robust estimation fusion problem in a distributed multi-sensor systems with bounded uncertainties, where a minimax robust estimation method was provided with the weighting matrices formulated as a solution of a semidefinite programming. Nevertheless, both [9] and [10] pay little attention to the performance analysis of the proposed algorithms. In [11], the author discussed the distributed robust estimation problem based on H_∞ consensus of estimates. To guarantee a certain level of H_∞ consensus, this paper provided a sufficient condition, which yet seems difficult to be verified. Although many robust filters have been provided, the stability of the algorithms still need further investigation under the distributed strategy.

The filter design plays an essential role in the study of distributed estimation problem over networks, since it directly influences the properties of the proposed algorithm, such as convergence of estimation error, boundedness of mean square error matrix, estimation precision, etc. [12] studied the distributed filter with constant filtering gain and fusion weight, based on which the relationship between the instability of system and the boundedness of estimation error was well analyzed. [13] proposed a measurement based distributed Kalman filter (DKF) and provided a design method on the time-invariant consensus weights. [14] studied a general diffusion DKF with constant weights and analyzed the performance of the proposed distributed filter based on a local observability assumption, which yet is strict for large-scale networks. In [15], [16], the design of time-varying parameters relied on the assumption that each agent knows the statistics of non-neighbors, which is not easy to be satisfied over a large network.

In this paper, we consider the distributed filtering problem with multiplicative noises in the system. Based on a general distributed filtering structure, adaptive fusion matrices with distributed strategy is considered in terms of consistency, which provides an upper bound for mean square error matrix of each agent and the bound is minimized by the designed time-varying filter gain. Under the design of adaptive filtering gain and weight matrix, a fully distributed Kalman filter without the requirement of global knowledge of the system [15], [16], [17] is provided. The proposed filter shows good robustness in coping multiplicative noises. Under mild conditions of collective observability and network topology,

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it is shown that the mean square error matrices of agents are uniformly upper bounded and the information matrix of the network are uniformly lower bounded by constant matrices both through finite time, which is revealed to be the summation of collective observability parameter and network scale. Additionally, it is shown that the proposed algorithm is convergent in the sense of expectation (i.e., asymptotic unbiasedness).

The remainder of the paper is organized as follows: Section II is on the problem formulation. Section III is on the design of the filter. Section IV is the performance analysis on the proposed filter. Section V shows the numerical simulation. The conclusion of this paper is given in section VI. Some proofs in this paper are omitted due to the limitation of pages.

A. Notations

$\mathbb{R}^n$ stands for the set of n-dimensional real vectors. Also, $\mathbb{R}^1 \triangleq \mathbb{R}$. $\mathbb{Z}^+$ stands for the set of positive integers. The superscript "T" represents the transpose. The notation $A \geq B$ (or $A > B$), where A and B are both symmetric matrices, means that $A - B$ is a positive semidefinite (or positive definite) matrix. I_n stands for the identity matrix with n rows and n columns. $E\{x\}$ denotes the mathematical expectation of the stochastic variable x. $diag\{\cdot\}$ represents the diagonalization of scalar elements. $|\mathcal{V}|$ denotes the size of the set $\mathcal{V}$.

II. Problem Formulation

We consider the following stochastic system over a multi-agent network

$$\begin{cases} x_{k+1} = (A + F\epsilon_k)x_k + \omega_k, \\ y_{k,i} = (C_i + G_i\gamma_{k,i})x_k + v_{k,i}, i = 1, 2, \cdots, N, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $\omega_k \in \mathbb{R}^n$ is the zero-mean white process noise with positive definite covariance matrix Q, $\epsilon_k \in \mathbb{R}$ is the zero-mean white multiplicative noise with variance $\mu_k < \infty$, subject to $\sup_k \mu_k < \infty$. For agent i, $y_{k,i} \in \mathbb{R}^{m_i}$ is measurement vector, $\gamma_{k,i}$ is the zero-mean white multiplicative noise with variance λ_i, and $v_{k,i} \in \mathbb{R}^{m_i}$ is mean-zero white measurement noise with known positive definite covariance matrices R_i. The initial state x_0 is zero-mean with known covariance P_0. The random variable and vectors $\{\epsilon_k\}_{k=0}^{\infty}$, $\{\omega_k\}_{k=0}^{\infty}$, $\{\gamma_{k,i}\}_{k=0}^{\infty}$, $\{v_{k,i}\}_{k=0}^{\infty}$ are mutually independent, and also independent of the initial state x_0. A, F, G_i and C_i are matrices with appropriate dimensions. N is the agent number over the network. Different from [15] requiring some global knowledge of statistics, we simply assume that Q, μ_k, R_i, λ_i, P_0, A, F, C_i, and G_i are known to agent i.

The communication between agents in the multi-agent network is modeled as a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, which consists of the set of agents or nodes $\mathcal{V} = \{1, 2, \ldots, N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacent matrix $\mathcal{A} = [a_{i,j}]$. In the weighted adjacent matrix $\mathcal{A}$, all the elements are nonnegative, row stochastic and the diagonal elements are all positive, i.e., $a_{i,i} > 0, a_{i,j} \geq 0, \sum_{j\in\mathcal{V}} a_{i,j} = 1$. If $a_{i,j} > 0, j \neq i$, there is an edge $(i,j) \in \mathcal{E}$, which means node i can directly receive the information of node j, and node j is called the neighbor of node i. All the neighbors of node i can be represented by the set $\{j \in \mathcal{V} | (i,j) \in \mathcal{E}\} \triangleq \mathcal{N}_i^0$. Also, $\mathcal{N}_i^0 \bigcup \{i\} \triangleq \mathcal{N}_i$. $\mathcal{G}$ is called strongly connected if for any pair nodes (i_1, i_l), there exists a directed path from i_1 to i_l consisting of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{l-1}, i_l)$. An undirected graph $\mathcal{G}$ is simply called connected if it is strongly connected, and it has a double stochastic (row and column) adjacent matrix $\mathcal{A} = [a_{i,j}]$, which requires $\sum_{i\in\mathcal{V}} a_{i,j} = 1, \forall j \in \mathcal{V}$, besides row stochastic.

In this paper, we consider the following distributed filtering structure for agent i , $\forall i \in \mathcal{V}$. This structure mainly consists of three parts: time prediction, measurement update and local fusion.

$$\begin{cases} \bar{x}_{k,i} = A\hat{x}_{k-1,i}, \\ \phi_{k,i} = \bar{x}_{k,i} + K_{k,i}(y_{k,i} - C_i\bar{x}_{k,i}), \\ \hat{x}_{k,i} = \sum_{j\in\mathcal{N}_i} W_{k,i,j}\phi_{k,j}, \end{cases} \tag{2}$$

where $\bar{x}_{k,i}$, $\phi_{k,i}$ and $\hat{x}_{k,i}$ are the state prediction, state update and state estimate of agent i at the kth moment, respectively. $K_{k,i}$ is the filtering gain matrix and $W_{k,i,j}$ is the local fusion matrix.

The objective of this paper is to design adaptive $K_{k,i}$ and $W_{k,i,j}$, such that the stochastic dynamic in (1) is well estimated by the filtering structure in (2) against the multiplicative noises.

III. Filter design

In the distributed estimation problem, since the accessible information at each agent is limited, a practical design of $K_{k,i}$ and $W_{k,i,j}$ should follow totally distributed strategy, which means each agent only employs the information of itself and its neighbors.

Considering the mutual independence between ϵ_k, x_k and ω_k, one can easily obtain the next lemma, which will be helpful in the subsequent analysis.

Lemma 1: The mean square of the state (i.e., $\Pi_k = E\{x_k x_k^T\}$) can be derived through

$$\Pi_{k+1} = A\Pi_k A^T + \mu_k F\Pi_k F^T + Q. \tag{3}$$

Lemma 1 provides an iterative approach to calculate the mean square of the system state, through which one can evaluate the probability range of the state.

In adaptive filters, Kalman filter is the minimal variance estimator for the linear Gaussian systems, thanks to the adaptive filtering gain calculated through P_k, which stands for the mean square error matrix. However, due to the strong correlation between estimates of agents over networks, it is difficult to derive the mean square error matrix through distributed strategy. Since the mean square error of each agent is not accessible, its upper bound is considered in this paper in terms of consistency, which is defined in the following.

Definition 1: ([18]) Suppose x_k is a random vector. Let $\hat{x}_k$ and P_k be the estimate of x_k and the estimate of the corresponding error variance matrix. Then the pair $(\hat{x}_k, P_k)$ is said to be consistent (or of consistency) at the kth moment if

$$E\{(\hat{x}_k - x_k)(\hat{x}_k - x_k)^T\} \leq P_k.$$

The following condition on the initial estimate of each agent is assumed for convenience. Note that Assumption 1 is easily satisfied and it can be guaranteed through setting a sufficient large $P_{0,i}$ for each agent.

Assumption 1: The initial condition satisfies $E\{(\hat{x}_{0,i} - x_0)(\hat{x}_{0,i} - x_0)^T\} \leq P_{0,i}$, $\forall i \in \mathcal{V}$.

The following theorem provides an adaptive design for the local fusion matrices $W_{k,i,j}$ in filtering structure (2), based on which the consistency of estimation $(\hat{x}_{k,i}, P_{k,i})$ is guaranteed.

Lemma 2: Considering the filtering structure (2), under Assumption 1, if

$$W_{k,i,j} = a_{i,j} P_{k,i} \tilde{P}_{k,j}^{-1}, \tag{4}$$

then the pairs

$$(\bar{x}_{k,i}, \bar{P}_{k,i}), (\phi_{k,i}, \tilde{P}_{k,i}), (\hat{x}_{k,i}, P_{k,i})$$

are all consistent, where $\bar{P}_{k,i}$, $\tilde{P}_{k,i}$ and $P_{k,i}$ are derived through

$$\begin{aligned} \bar{P}_{k,i} =& A P_{k-1,i} A^T + \mu_{k-1} F \Pi_{k-1} F^T + Q, && (5)\\ \tilde{P}_{k,i} =& (I - K_{k,i} C_i) \bar{P}_{k,i} (I - K_{k,i} C_i)^T \\ & + K_{k,i} \bar{R}_{k,i} K_{k,i}^T && (6)\\ P_{k,i} =& (\sum_{j \in \mathcal{N}_i} a_{i,j} \tilde{P}_{k,j}^{-1})^{-1}. && (7) \end{aligned}$$

where $\bar{R}_{k,i} = \lambda_i G_i \Pi_k G_i^T + R_i$.

Proof: At the initial value, under Assumption 1, we have

$$E\{(\hat{x}_{0,i} - x_0)(\hat{x}_{0,i} - x_0)^T\} \leq P_{0,i}. \tag{8}$$

To finish the proof, we use the inductive method. Suppose at the kth time instant,

$$E\{(\hat{x}_{k,i} - x_k)(\hat{x}_{k,i} - x_k)^T\} \leq P_{k,i}. \tag{9}$$

In the prediction step, since $\omega_k, e_{k,i}$ and ϵ_k are independent of each other, it can be derived that

$$\begin{aligned} & E\{(\bar{x}_{k+1,i} - x_{k+1})(\bar{x}_{k+1,i} - x_{k+1})^T\} \\ =& E\{(A e_{k,i} - \epsilon_k F x_k - \omega_k)(A e_{k,i} - \epsilon_k F x_k - \omega_k)^T\} \\ =& A E\{e_{k,i} e_{k,i}^T\} A^T + \mu_k F E\{x_k x_k^T\} F^T + Q \\ \leq& A P_{k,i} A^T + \mu_k F \Pi_k F^T + Q \\ =& \bar{P}_{k+1,i}. \end{aligned} \tag{10}$$

In the filtering update step, because of the mutual independence between $\bar{e}_{k+1}$ and $v_{k+1,i}$, one can obtain

$$\begin{aligned} & E\{(\phi_{k+1,i} - x_{k+1})(\phi_{k+1,i} - x_{k+1})^T\} \\ =& E\{\tilde{e}_{k+1,i} \tilde{e}_{k+1,i}^T\} \\ \leq& (I_n - K_{k+1,i} C_i) E\{\bar{e}_{k+1} \bar{e}_{k+1}^T\} (I_n - K_{k+1,i} C_i)^T \\ & + K_{k+1,i} (\lambda_i G_i \Pi_k G_i^T + R_i) K_{k+1,i}^T \\ \leq& (I_n - K_{k+1,i} C_i) \bar{P}_{k+1} (I_n - K_{k+1,i} C_i)^T \\ & + K_{k+1,i} (\lambda_i G_i \Pi_{k+1} G_i^T + R_i) K_{k+1,i}^T \\ =& \tilde{P}_{k+1}. \end{aligned} \tag{11}$$

In the local fusion step, under the design of $W_{k,i,j}$ in (4), according to (11) and the consistent estimation of CI strategy ([19]), we have

$$E\{(\hat{x}_{k+1,i} - x_{k+1})(\hat{x}_{k+1,i} - x_{k+1})^T\} \leq P_{k+1,i}. \tag{12}$$

According to (9) – (12), under the initial condition (8), it is safe to obtain the conclusions of this theorem. ■

Lemma 3: Minimizing $\tilde{P}_{k,i}$ (6) in the sense of positive definiteness with respect to the gain matrix $K_{k,i}$ yields

$$K_{k,i} = \bar{P}_{k,i} C_i^T (C_i \bar{P}_{k,i} C_i^T + \bar{R}_{k,i})^{-1}. \tag{13}$$

where $\bar{R}_{k,i} = \lambda_i G_i \Pi_k G_i^T + R_i$. Also, (6) is equivalent to

$$\tilde{P}_{k,i} = (I - K_{k,i} C_i) \bar{P}_{k,i}. \tag{14}$$

Proof: Consider (6), then we have

$$\begin{aligned} \tilde{P}_{k,i} =& (I_n - K_{k,i} C_i) \bar{P}_{k,i} (I_n - K_{k,i} C_i)^T \\ & + K_{k,i} \bar{R}_{k,i} K_{k,i}^T \\ =& \bar{P}_{k,i} - K_{k,i} C_i \bar{P}_{k,i} - \bar{P}_{k,i} C_i^T K_{k,i}^T \\ & + K_{k,i} C_i \bar{P}_{k,i} C_i^T K_{k,i}^T + K_{k,i} \bar{R}_{k,i} K_{k,i}^T \\ =& \bar{P}_{k,i} - K_{k,i} C_i \bar{P}_{k,i} - \bar{P}_{k,i} C_i^T K_{k,i}^T \\ & + K_{k,i} (C_i \bar{P}_{k,i} C_i^T + \bar{R}_{k,i}) K_{k,i}^T \\ =& (K_{k,i} - K_{k,i}^*)(C_i \bar{P}_{k,i} C_i^T + R_i)(K_{k,i} - K_{k,i}^*)^T \\ & + (I - K_{k,i}^* C_i) \bar{P}_{k,i}, \end{aligned} \tag{15}$$

where $K_{k,i}^* = \bar{P}_{k,i} C_i^T (C_i \bar{P}_{k,i} C_i^T + \bar{R}_{k,i})^{-1}$.

Thus, from (15) it is seen that $\tilde{P}_{k,i}$ is minimized in the sense of positive definiteness when $K_{k,i} = K_{k,i}^*$. ■

Summing up Lemmas 2 and 3, the distributed robust Kalman filter (DRKF) based on filtering structure (2) is proposed in Table I.

IV. Performance analysis

In the subsequent performance analysis, the following assumptions are needed.

Assumption 2: The directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of the multi-agent network is strongly connected.

Assumption 2 is a general condition in the study of distributed estimation over a multi-agent network [20], [21]. If the network is completely connected [15], this assumption can be naturally satisfied.

TABLE I
DISTRIBUTED ROBUST KALMAN FILTER (DRKF):

Prediction:
$\bar{x}_{k,i} = A\hat{x}_{k-1,i}$,
$\bar{P}_{k,i} = AP_{k-1,i}A^T + \mu_{k-1}F\Pi_{k-1}F^T + Q$,
Measurement Update:
$\phi_{k,i} = \bar{x}_{k,i} + K_{k,i}(y_{k,i} - C_i\bar{x}_{k,i})$
$K_k = \bar{P}_{k,i}C_i^T\left(C_i\bar{P}_{k,i}C_i^T + \lambda_i G_i \Pi_k G_i^T + R_i\right)^{-1}$
$\tilde{P}_{k,i} = (I - K_kC_i)\bar{P}_{k,i}$,
Local Fusion: Receiving ($\phi_{k,j}$, $\tilde{P}_{k,j}$) from neighbors $j \in \mathcal{N}_i$
$\hat{x}_{k,i} = P_{k,i}\sum_{j\in\mathcal{N}_i} a_{i,j}\tilde{P}_{k,j}^{-1}\phi_{k,j}$,
$P_{k,i} = (\sum_{j\in\mathcal{N}_i} a_{i,j}\tilde{P}_{k,j}^{-1})^{-1}$.

Assumption 3: There exists an integer $\bar{N} > 0$ and a scalar $\alpha > 0$, such that for $\forall k \in \mathbb{Z}^+$,

$$\sum_{i=1}^{N}\sum_{j=k}^{k+\bar{N}}(A^{k-1})^T C_i^T \bar{R}_{j,i}^{-1} C_i A^{k-1} > \alpha I_n, \tag{16}$$

where $\bar{R}_{j,i} = \lambda_i G_i \Pi_j G_i^T + R_i$.

Assumption 3 is a very mild requirement for the system with multiplicative noises over measurements, since it permits the local observation condition of each agent is not met. If there is no multiplicative noise over measurements, then Assumption 3 degenerates to the condition that (A, C) is observable, where $C = [C_1^T, C_2^T, \cdots, C_N^T]^T$, which is a general requirement for distributed estimation [20], [21], [22]. As is known, Assumption 3 is also a necessary condition to guarantee the stability of centralized Kaman filter which is optimal in some sense. It is noted that, in a single filter, the multiplicative random noise of the system (i.e., process equation and measurement equation) will lead to an irretrievable loss, if the state of the system is divergent [23], [15]. However, in the distributed estimation, it is shown in (16) that measurements of some agents are allowed to tolerate the multiplicative noise.

Definition 2: The integer $\bar{N}$ satisfying (16) is called the collective observability parameter.

Assumption 4: $\sup_k\{\mu_k F\Pi_k F^T\} < \infty$.

In the absence of multiplicative noises (i.e., $\mu_k = 0, k \in \mathbb{Z}^+$), Assumption 4 holds naturally. In the presence of multiplicative noises (i.e., $\mu_k > 0$), Assumption 4 can also be satisfied not only for bounded covariance matrices of multiplicative noise and state [15], [17], but also for the unbounded ones [20] if the speed of μ_k converging to zero is fast enough. It is noted that the condition is reasonable, since a rather large noise may lead to an irretrievable loss for the state estimation.

Lemma 4: A sufficient condition to satisfy Assumption 4 is

$$\|A\|_2^2 + \mu_k\|F\|_2^2 < 1, \forall k \in \mathbb{Z}^+. \tag{17}$$

Proof: From (3), we have

$$\begin{aligned}\|\Pi_{k+1}\|_2 &= \|A\Pi_k A^T + \mu_k F\Pi_k F^T + Q\|_2 \\ &\le \|\Pi_k\|_2(\|A\|_2^2 + \mu_k\|F\|_2^2) + \|Q\|_2.\end{aligned} \tag{18}$$

In light of (17), by induction, $\|\Pi_{k+1}\|_2$ is uniformly upper bounded. Considering $\sup_k \mu_k < \infty$, Assumption 4 can be satisfied. ■

Remark 1: In the spatio-temporal applications ([24]), if the correlation function of data is stationary, the corresponding state-space form provides a stable system matrix A (i.e., $\|A\|_2 < 1$).

The following theorem provides the upper boundedness of mean square error matrix at each agent.

Theorem 1: Under Assumptions 1–4, if A is non-singular, then there exists a constant matrix $\breve{P}$, such that for $k \ge L(L \triangleq N + \bar{N})$,

$$E\{(\hat{x}_{k,i} - x_k)(\hat{x}_{k,i} - x_k)^T\} \le P_{k,i} \le \breve{P}, \forall i \in \mathcal{V}. \tag{19}$$

Remark 2: It is seen from Theorem 1 that the mean square error matrix of each agent is uniformly bounded through finite time (i.e., $N+\bar{N}$), which is described by the collective observability parameter (i.e., $\bar{N}$) and the scale of the network (i.e., N).

The following conclusion illustrates the information accumulation of the network by using the proposed algorithm.

Corollary 1: Under the same condition as Theorem 1, and if the network is undirected, then there exists $\beta > 0$, such that

$$I_k^* = \sum_{i\in\mathcal{V}} I_{k,i}^* \ge \beta I_n > 0,$$

where $I_{k,i}^* = (E\{(\hat{x}_{k,i} - x_k)(\hat{x}_{k,i} - x_k)^T\})^{-1}$ and $k \ge N + \bar{N}$.

It is seen from Corollary 1 that through finite time (i.e., $N + \bar{N}$), the information matrix I_k^* of the network will be lower bounded by a constant information matrix, which is independent of the concrete form of the adjacent matrix and simply related with the system matrices. In the view of information theory, this property illustrates that the proposed algorithm can make full use of the system information regardless of the adjacent matrix artificially designed.

Theorem 1 shows that the mean square error of each agent is eventually upper bounded. The asymptotic unbiasedness of estimation is provided in the following.

Theorem 2: Under the same conditions as Theorem 1, the estimates of all the agents over the network using the proposed DRKF are asymptotic unbiased (or convergent in expectation), i.e.,

$$\lim_{k\to+\infty} E\{\hat{x}_{k,i} - x_k\} = 0, \quad \forall i \in \mathcal{V}. \tag{20}$$

Theorem 2 reveals that under the disturbances of multiplicative noises, the proposed DRKF can still guarantee the asymptotic convergence in expectation. Combining the conclusions of Theorems 2 and 1, one can not only ensure the convergence of the algorithm before its implementation but also evaluate the estimation precision of the filter at real time.

V. NUMERICAL SIMULATION

In this section, we study a numerical simulation to test the effectiveness of the proposed DRKF algorithm in Table I.

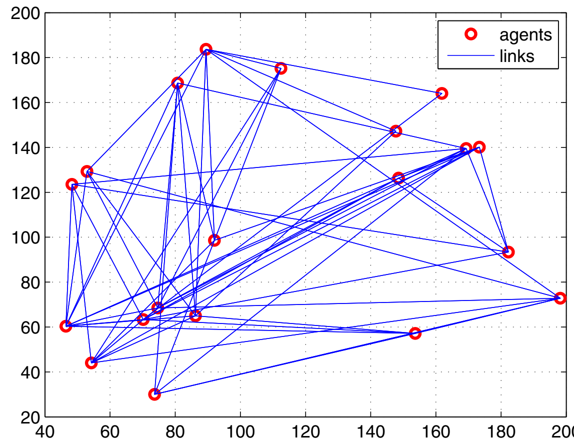

Fig. 1. Communication topology of a agent network with 20 agents

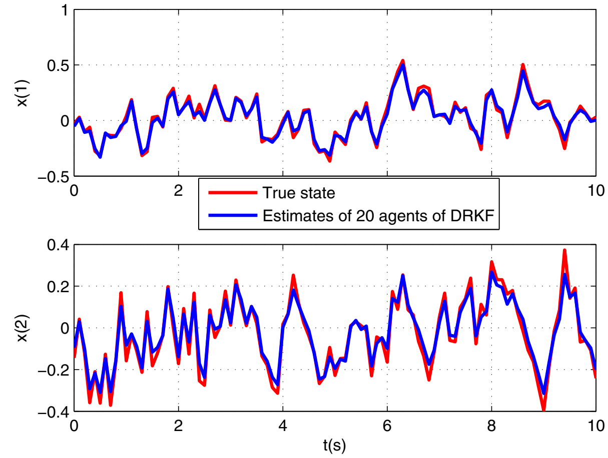

Fig. 2. Tracking performance of distributed filter with DRKF

Consider the following second-order stochastic dynamics observed by twenty agents over a undirected and connected network, which is illustrated in Fig. 1.

$$\begin{cases} x_{k+1} = \begin{pmatrix} 0.6+\epsilon_k & 0.12 \\ 0.1 & 0.5+\epsilon_k \end{pmatrix} x_k + \omega_k, \\ y_{k,i} = (C_i + [1,1]\gamma_{k,i})x_k + v_{k,i}, i = 1, \cdots, 20, \end{cases} \quad (21)$$

where the observation matrices of the twenty agents over the network are uniformly randomly selected from the following matrix set $\{(2,1),(1,0),(0,1),(1,2)\}$, which corresponds to the variance set of $\gamma_{k,i}$: $\{0.01, 0.02, 0.05, 0.07\}$.

Here, it is assumed that the process noise covariance matrix $Q = diag\{10, 10\}$, and the whole measurement noise covariance matrix $R_i = 1, i \in \mathcal{V}$. The initial value of the state is generated by a Gaussian process with zero mean and covariance matrix I_2, and the initial estimation settings are $\hat{x}_{i,0} = 0$ and $P_{i,0} = diag\{10, 10\}$, $\forall i \in \mathcal{V}$. The weighted adjacent matrix $\mathcal{A} = [a_{i,j}]$ is designed as $a_{i,j} = \frac{1}{|\mathcal{N}_i|}, j \in \mathcal{N}_i, i,j \in \mathcal{V}$. We conduct the numerical simulation through Monte Carlo experiment, in which 500 Monte Carlo trials are performed. The mean square error of the whole network is defined as

$$MSE_k = \frac{1}{20}\sum_{i=1}^{20}\frac{1}{500}\sum_{j=1}^{500}(\hat{x}_{k,i}^j - x_k^j)^T(\hat{x}_{k,i}^j - x_k^j),$$

where $\hat{x}_{k,i}^j$ is the state estimation of the jth trail of agent i at the kth moment.

The numerical simulation is carried out for the considered system in (21) through employing the proposed distributed robust Kalman filter. The tracking result with respect to the proposed DRKF is provided in Fig. 2. From Fig. 2, it can be seen that the stochastic dynamics can be well estimated with DRKF by agents over the network. Further, the mean square estimation error is shown in Fig. 3, which shows that the estimation error of DRKF keeps stable and the consistency of the proposed DRKF remains, i.e., an upper bound can be obtained in real time to evaluate the estimation performance.

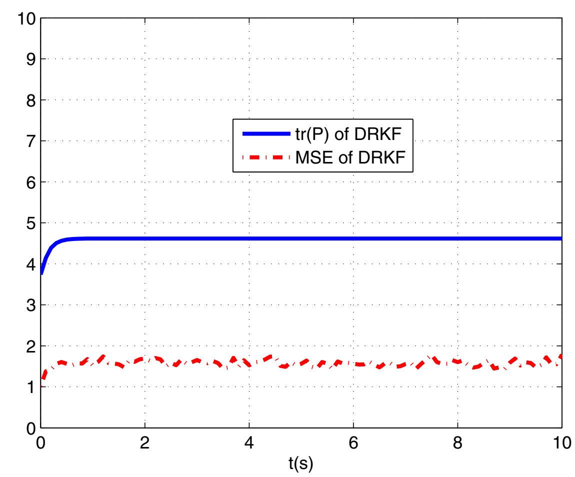

Fig. 3. Tracking performance comparison between different filters

The above results reveal that the proposed DRKF is an effective and flexible distributed state estimation algorithm.

VI. Conclusion

This paper studied the distributed state estimation problem for the LTI systems with multiplicative noises. An online adaptive design of filtering gain and fusion matrices without requiring any global knowledge was provided. Under connectivity of network and collective observability conditions of the system, it was shown that the mean square error matrices of agents are uniformly upper bounded and the information matrix of the network are uniformly lower bounded by constant matrices both through finite time, which is revealed to be the summation of collective observability parameter and network scale. The convergence of the estimation was guaranteed in expectation.

References

[1] Y. Sun, M. Fu, B. Wang, H. Zhang, and D. Marelli, "Dynamic state estimation for power networks using distributed MAP technique," *Automatica*, vol. 73, no. 11, pp. 27–37, 2016.

[2] Z. Zhou, H. Fang, and Y. Hong, “Distributed estimation for moving target based on state-consensus strategy,” *IEEE Transactions on Automatic Control*, vol. 58, no. 8, pp. 2096–2101, 2013.
[3] W. Yang, C. Yang, H. Shi, L. Shi, and G. Chen, “Stochastic link activation for distributed filtering under sensor power constraint,” *Automatica*, vol. 75, pp. 109–118, 2017.
[4] X. Lu, L. Xie, H. Zhang, and W. Wang, “Robust Kalman filtering for discrete-time systems with measurement delay,” *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 54, no. 6, pp. 522–526, 2007.
[5] L. Zhang, Z. Ning, and Z. Wang, “Distributed filtering for fuzzy time-delay systems with packet dropouts and redundant channels,” *IEEE Transactions on Systems Man & Cybernetics Systems*, vol. 46, no. 4, pp. 559–572, 2016.
[6] B. S. Chow and W. Birkemeier, “A new recursive filter for systems with multiplicative noise,” *IEEE transactions on information theory*, vol. 36, no. 6, pp. 1430–1435, 1990.
[7] F. Yang, Z. Wang, and Y. Hung, “Robust Kalman filtering for discrete time-varying uncertain systems with multiplicative noises,” *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1179–1183, 2002.
[8] T. Zhou, “Robust recursive state estimation with random measurement droppings,” *IEEE Transactions on Automatic Control*, vol. 61, no. 1, pp. 156–171, 2016.
[9] J. Feng, Z. Wang, and M. Zeng, “Distributed weighted robust Kalman filter fusion for uncertain systems with autocorrelated and cross-correlated noises,” *Information Fusion*, vol. 14, no. 1, pp. 78–86, 2013.
[10] X. Qu, J. Zhou, E. Song, and Y. Zhu, “Minimax robust optimal estimation fusion in distributed multisensor systems with uncertainties,” *IEEE Signal Processing Letters*, vol. 17, no. 9, pp. 811–814, 2010.
[11] V. Ugrinovskii, “Distributed robust filtering with H_∞ consensus of estimates,” *Automatica*, vol. 47, no. 1, pp. 1–13, 2011.
[12] U. A. Khan and A. Jadbabaie, “Collaborative scalar-gain estimators for potentially unstable social dynamics with limited communication,” *Automatica*, vol. 50, no. 7, pp. 1909–1914, 2014.
[13] S. Das and J. M. F. Moura, “Distributed Kalman filtering with dynamic observations consensus,” *Signal Processing IEEE Transactions on*, vol. 63, no. 17, pp. 4458–4473, 2015.
[14] F. S. Cattivelli and A. H. Sayed, “Diffusion strategies for distributed Kalman filtering and smoothing,” *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2069–2084, 2010.
[15] Y. Liu, Z. Wang, X. He, and D. Zhou, “Minimum-variance recursive filtering over sensor networks with stochastic sensor gain degradation: Algorithms and performance analysis,” *IEEE Transactions on Control of Network Systems*, vol. 3, no. 3, pp. 265–274, 2016.
[16] W. Li, Y. Jia, and J. Du, “Distributed filtering for discrete-time linear systems with fading measurements and time-correlated noise,” *Digital Signal Processing*, vol. 60, pp. 211–219, 2017.
[17] C. Wen, Z. Wang, Q. Liu, and F. E. Alsaadi, “Recursive distributed filtering for a class of state-saturated systems with fading measurements and quantization effects,” *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2016.
[18] S. J. Julier and J. K. Uhlmann, “A non-divergent estimation algorithm in the presence of unknown correlations,” vol. 4, pp. 2369–2373, 1997.
[19] W. Niehsen, “Information fusion based on fast covariance intersection filtering,” in *International Conference on Information Fusion*, pp. 901–904, 2002.
[20] X. He, W. Xue, and H. Fang, “Consistent distributed state estimation with global observability over sensor network,” *Automatica*, vol. 92, pp. 162–172, 2018.
[21] X. He, C. Hu, W. Xue, and H. Fang, “On event-based distributed kalman filter with information matrix triggers,” vol. 50, no. 1, p-p. 14308–14313, 2017.
[22] G. Battistelli and L. Chisci, “Kullback-cleibler average, consensus on probability densities, and distributed state estimation with guaranteed stability ,” *Automatica*, vol. 50, no. 3, pp. 707–718, 2014.
[23] J. Tugnait, “Stability of optimum linear estimators of stochastic signals in white multiplicative noise,” *IEEE Transactions on Automatic Control*, vol. 26, no. 3, pp. 757–761, 1981.
[24] S. Sarkka, A. Solin, and J. Hartikainen, “Spatiotemporal learning via infinite-dimensional bayesian filtering and smoothing: A look at gaussian process regression through Kalman filtering,” *IEEE Signal Processing Magazine*, vol. 30, no. 4, pp. 51–61, 2013.