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Concurrent optimal control and actuator design for semi-linear systems

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***Abstract*— Semi-linear partial differential equations model a wide spectrum of physical systems with distributed parameters. It is shown that under sufficient conditions on nonlinearities in the systems and in cost functions, an optimal control input and optimal actuator design exist. First-order optimality conditions are obtained that characterize the optimizer. The results are used to address vibration suppression in a nonlinear railway track.**

I. Introduction

Finding the best actuator location to control a distributed parameter system can improve performance and significantly reduce the cost of the control; see for example [1]. The optimal actuator location problem has been discussed by many researchers in various contexts [2], [3]. From a theoretical point of view, the existence of an optimal actuator location has been discussed in the literature for linear partial differential equations (PDEs). In [4], it is was proven that an optimal actuator location exists for a linear-quadratic control system with a compact input operator that continuously depends on actuator locations. From a practical point of view, further conditions on operators and cost functions are needed to guarantee the convergence in numerical schemes [4]. Similar results have been obtained for H_2 and H_∞ controller design objectives [5], [6].

Nonlinearities in some distributed parameter systems have a significant effect on their dynamics, and such systems cannot be accurately modeled by linear models. Optimal control has been studied for a number of applications, including wastewater treatment systems [7], steel cooling plants [8], oil extraction through a reservoir [9], solidification models in metallic alloys [10], thermistors [11], the Schlögl model [12], static elastoplasticity [13], and the Fokker-Planck equation [14].

A review of PDE-constrained optimization theory can be found in the books [15], [16] in which a variety of optimization problems constrained by PDEs are discussed. In [17], [18], first-order optimality conditions are investigated for parabolic partial differential equations.

Hence, a control design and actuator location strategy should take the nonlinear behavior into consideration. There are a few studies in the engineering literature on optimal actuator locations in nonlinear systems; see for example, [19], [20]. Theory for the concurrent optimal control and actuator design of a class of controlled semi-linear PDEs is described in this paper. The research described extends previous work in that the linear part of the partial differential equation is not constrained to be the generator of an analytic semigroup. A general class of PDEs with weakly continuous nonlinear part is considered. It is shown that the problem has an optimal control and actuator design. Under additional assumptions, optimality equations explicitly characterizing the optimal control and actuator are obtained. Details of the work described in this talk can be found in [21].

Vibration control of nonlinear beam models as well as optimal actuator location in flexible structures have attracted attention in recent years; see for example [22], [23], [1], [24]. The result of this study is applied to a semi-linear beam model. This model predicts the dynamic behavior of railway tracks and its underneath foundation. This application is primarily motivated by the need for an optimal control strategy in the vibration suppression of railway tracks [25]. The behaviour of the foundation introduces nonlinearity into the model.

II. Main Results

Consider a semi-linear system with state $\mathfrak{z}(t)$ on a separable reflexive Banach state space $\mathbb{Z}$:

$$\dot{\mathfrak{z}}(t) = \mathcal{A}\mathfrak{z}(t) + \mathcal{F}(\mathfrak{z}(t)) + \mathcal{B}(r)u(t), \quad \mathfrak{z}(0) = \mathfrak{z}_0 \in \mathbb{Z}, \quad (1)$$

The function $u(t)$ is the input to the system, and takes values in a Banach space $\mathbb{U}$. The control operator $\mathcal{B}$ depends on a parameter r that takes values in a set K_{ad}. The parameter r typically has interpretation as possible actuator locations. The operators $\mathcal{A}$, $\mathcal{F}(\cdot)$, and $\mathcal{B}$ satisfy the following assumptions.

Assumption A.

1) *The state operator $\mathcal{A}$ with domain $D(\mathcal{A})$ generates a strongly continuous semigroup $\mathcal{T}(t)$ on $\mathbb{Z}$.*
2) *The nonlinear operator $\mathcal{F}(\cdot)$ is weakly sequentially continuous, i.e., if $\mathfrak{z}_n \rightharpoonup \mathfrak{z}$, then $\mathcal{F}(\mathfrak{z}_n) \rightharpoonup \mathcal{F}(\mathfrak{z})$ on $\mathbb{Z}$. It is also locally Lipschitz continuous on $\mathbb{Z}$; that is, for every positive number δ, there exists $L_{\mathcal{F}\delta} > 0$ such that*
$$\|\mathcal{F}(\mathfrak{z}_2) - \mathcal{F}(\mathfrak{z}_1)\| \le L_{\mathcal{F}\delta} \|\mathfrak{z}_2 - \mathfrak{z}_1\|,$$
for all $\|\mathfrak{z}_2\| \le \delta$ and $\|\mathfrak{z}_1\| \le \delta$.
3) *For each $r \in K_{ad}$, the input operator $\mathcal{B}(r)$ is a linear bounded operator that maps the input space $\mathbb{U}$ into the state space $\mathbb{Z}$. This family of operators is uniformly bounded over K_{ad}, i.e., there exist a positive number $M_{\mathcal{B}}$ such that $\|\mathcal{B}(r)\| \le M_{\mathcal{B}}$ for all $r \in K_{ad}$.*

In some cases, due to lack of regularity of the input u a classical solution to the IVP (1) is not assured. With these

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assumptions, the existence of a unique mild solution to the initial value problem (IVP) (1) is proven assuming only that $u \in L^p(0,T;\mathbb{U})$, $1 < p < \infty$.

Theorem 1. *Under Assumption A, for each $\mathfrak{z}_0 \in \mathbb{Z}$ and positive number R, there exists $T > 0$ such that the IVP (1) admits a unique local mild solution $\mathfrak{z} \in C(0,T;\mathbb{Z})$ for all $u \in L^p(0,T;\mathbb{U})$, $\|u\|_p \le R$, and all $r \in K_{ad}$.*

For functionals $\phi(\mathfrak{z})$ on $\mathbb{Z}$ and $\psi(u)$ on $\mathbb{U}$, consider the cost function

$$J(u,r;\mathfrak{z}_0) = \int_0^T \phi(\mathfrak{z}(t)) + \psi(u(t))\,dt, \tag{2}$$

where the admissible control input $u(t)$ belongs to the set

$$U_{ad} = \{u \in L^p(0,T;\mathbb{U}) |\ \|u\|_p \le R\}.$$

The optimization problem is to minimize $J(u,r;\mathfrak{z}_0)$ over all admissible control inputs $u \in U_{ad}$, and also over all admissible actuator locations $r \in K_{ad}$, subject to the IVP (1) with a fixed initial condition $\mathfrak{z}_0 \in \mathbb{Z}$. That is,

$$\begin{cases} \min & J(u,r;\mathfrak{z}_0) \\ \text{s.t.} & \dot{\mathfrak{z}}(t) = \mathcal{A}\mathfrak{z}(t) + \mathcal{F}(\mathfrak{z}(t)) + \mathcal{B}(r)u(t), \quad \forall t \in (0,T] \\ & \mathfrak{z}(0) = \mathfrak{z}_0, \\ & u \in U_{ad} \\ & r \in K_{ad} \end{cases} \tag{P}$$

To guarantee the existence of a unique optimizer, further assumptions are needed on the operators $\mathcal{B}(r)$, the set K_{ad}, and the cost function $J(u,r;\mathfrak{z}_0)$.

Assumption B.

1) *Let K_{ad} be a compact set in the actuator location space $\mathbb{K}$. The family of input operators $\mathcal{B}(\cdot) : K_{ad}(\subset \mathbb{K}) \to \mathcal{L}(\mathbb{U},\mathbb{Z})$ are continuous with respect to r in the operator norm topology:*

$$\lim_{r_2 \to r_1} \|\mathcal{B}(r_2) - \mathcal{B}(r_1)\| = 0.$$

2) *The functionals $\phi(\cdot)$ and $\psi(\cdot)$ are weakly lower semi-continuous positive functionals on $\mathbb{Z}$ and $\mathbb{U}$, respectively.*

It is shown that under these assumptions, an optimal control and actuator exist.

Theorem 2. *For initial condition $\mathfrak{z}_0 \in \mathbb{Z}$ let T be such that the mild solution exists for all $u \in U_{ad}$, and all $r \in K_{ad}$. Under Assumptions A and B, there exists a control input $u^o \in U_{ad}$ together with an actuator location $r^o \in K_{ad}$, that solve the optimization problem P.*

To characterize an optimizer to the optimization problem, further assumptions on differentiability of the nonlinear operator $\mathcal{F}$ and the cost function are needed.

Assumption C.

1) *The nonlinear operator $\mathcal{F}(\cdot)$ is Fréchet differentiable on $\mathbb{Z}$. Indicate the Fréchet derivative of $\mathcal{F}(\cdot)$ at $\mathfrak{z}$ by $\mathcal{F}'_{\mathfrak{z}}$.*
2) *The mapping $\mathfrak{z} \mapsto \mathcal{F}'_{\mathfrak{z}}$ is bounded, i.e., bounded sets in $\mathbb{Z}$ are mapped into bounded sets in $\mathcal{L}(\mathbb{Z})$.*
3) *The control operator $\mathcal{B}(r)$ is Fréchet differentiable with respect to r from $\mathbb{K}$ to $\mathcal{L}(\mathbb{U},\mathbb{Z})$. Indicate the Fréchet derivative of $\mathcal{B}(r)$ at r by $\mathcal{B}'_r$.*
4) *The state space $\mathbb{Z}$ is a Hilbert space and the input space $\mathbb{U}$, and actuator location space $\mathbb{K}$ are each finite-dimensional Hilbert spaces.*
5) *In the cost function (2), consider the functionals*

$$\phi(\mathfrak{z}) = \langle \mathcal{Q}\mathfrak{z}, \mathfrak{z} \rangle, \quad \psi(u) = \langle \mathcal{R}u, u \rangle_{\mathbb{U}},$$

where the linear operator $\mathcal{Q}$ is a positive semi-definite, self-adjoint bounded operator on $\mathbb{Z}$, and the linear operator $\mathcal{R}$ is a positive definite, self-adjoint bounded operator on $\mathbb{U}$.

In the following theorem, the mapping $\mathcal{S}(u;r,\mathfrak{z}_0)$ yields a state $\mathfrak{z} \in C(0,T;\mathbb{H})$ as a solution to IVP (1) with an input $u \in L^2(0,T;\mathbb{U})$, actuator location $r \in \mathbb{K}$, and initial condition $\mathfrak{z}_0 \in \mathbb{H}$.

Theorem 3. *Suppose that Assumptions A, B, and C hold. For every initial condition $\mathfrak{z}_0 \in \mathbb{Z}$, let the pair (u^o, r^o) be a local minimizer of the optimization problem P with the optimal trajectory $\mathfrak{z}^o = \mathcal{S}(u^o; r^o, \mathfrak{z}_0)$. Set $u = (u_1, u_1, ..., u_n) \in \mathbb{R}^n$ and $r = (r_1, r_1, ..., r_m) \in \mathbb{R}^m$; this implies that*

$$(\mathcal{B}_{r^o} r)u = \sum_{j=1}^{m}\sum_{i=1}^{n} \frac{\partial b_i}{\partial r_j}(r^o) r_j u_i,$$

for some differentiable $\mathbb{Z}$-valued functions $b_i(\cdot)$. If (u^o, r^o) is an interior point of $U_{ad} \times K_{ad}$ then (u^o, r^o) satisfies

$$u^o(t) = -\mathcal{R}^{-1}\mathcal{B}^*(r^o)\mathfrak{p}^o(t), \tag{3}$$

$$\sum_{i=1}^{n} \int_0^T \left\langle \mathfrak{p}^o(s), \frac{\partial b_i}{\partial r_j}(r^o) \right\rangle u_i^o(s) ds = 0, \quad j = 1, 2, ..., m,$$

where $\mathfrak{p}^o(t)$, the adjoint state, is the mild solution of the final value problem (FVP):

$$\dot{\mathfrak{p}}^o(t) = -(\mathcal{A}^* + \mathcal{F}'^*_{\mathfrak{z}^o(t)})\mathfrak{p}^o(t) - \mathcal{Q}\mathfrak{z}^o(t), \quad \mathfrak{p}^o(T) = 0.$$

III. Nonlinear Railway Track Model

Railway tracks are rested on ballast which is known for exhibiting nonlinear viscoelastic behavior [26]. If a track beam is made of a Kelvin-Voigt material, then the railway track model will be a semi-linear partial differential equation on $x \in [0, \ell]$ as follows:

$$\rho a \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2} + C_d \frac{\partial^3 w}{\partial x^2 \partial t}\right) + \mu \frac{\partial w}{\partial t} + kw + \alpha w^3 = b(x;r)u(t), \tag{4}$$

$$w(x,0) = w_0(x), \quad \frac{\partial w}{\partial t}(x,0) = v_0(x),$$

$$w(0,t) = w(\ell,t) = 0,$$

$$EI\frac{\partial^2 w}{\partial x^2}(0,t) + C_d \frac{\partial^3 w}{\partial x^2 \partial t}(0,t) = 0.$$

$$EI\frac{\partial^2 w}{\partial x^2}(\ell,t) + C_d \frac{\partial^3 w}{\partial x^2 \partial t}(\ell,t) = 0,$$

where the positive constants E, I, ρ, a, and ℓ are the modulus of elasticity, second moment of inertia, density of the beam, cross-sectional area, and length of the beam, respectively. The linear and nonlinear parts of the foundation elasticity correspond to the coefficients k and α, respectively. The constant $\mu \geq 0$ is the damping coefficient of the foundation, and $C_d \geq 0$ is the coefficient of Kelvin-Voigt damping in the beam. The external force exerted on the railway track is denoted by $u(t)$. It will be considered as a scalar control input to manipulate the system. The function $b(x;r)$ is a piece-wise continuous function in x parametrized by r, the actuator locations. The function $b(x;r)$ will describe the effect of actuators on the system. It is assumed to be a sufficiently smooth function of the actuator location so that Assumptions B1 and C3 are satisfied.

Define the closed self-adjoint positive operator $\mathcal{A}_0$ on $L^2(0,\ell)$ as:

$$\begin{aligned} \mathcal{A}_0 w &:= w_{xxxx}, \\ D(\mathcal{A}_0) &:= \{w \in H^4(0,\ell) |\, w(0) = w(\ell) = 0, \\ &\qquad w_{xx}(0) = w_{xx}(\ell) = 0\}, \end{aligned} \tag{5}$$

where subscripts denote the derivative with respect to spatial variable. Consider the state space $\mathbb{Z} = H^2(0,\ell) \cap H_0^1(0,\ell) \times L^2(0,\ell)$ with the inner product

$$\langle (w_1, v_1), (w_2, v_2) \rangle = \int_0^L EI w_{1xx} \bar{w}_{2xx} + k w_1 \bar{w}_2 + \rho a v_1 \bar{v}_2 \, dx. \tag{6}$$

The state operator associated with the Kelvin-Voigt beam is

$$\mathcal{A}_{KV}(w,v) := \left(v, -\frac{1}{\rho a} \mathcal{A}_0 (EIw + C_d v) \right), \tag{7}$$

with domain

$$\begin{aligned} D(\mathcal{A}_{KV}) &:= \{(w,v) \in \mathbb{Z} |\, v \in H^2(0,\ell) \cap H_0^1(0,\ell), \\ &\qquad EIw + C_d v \in D(\mathcal{A}_0)\}, \end{aligned} \tag{8}$$

which is dense in $\mathbb{Z}$. The state space $\mathbb{Z}$ is separable since the spaces $H^2(0,\ell) \cap H_0^1(0,\ell)$ and $L^2(0,\ell)$ are separable. Furthermore, define the linear operators $\mathcal{K}$, $\mathcal{B}(r)$, and the nonlinear operator $\mathcal{F}(\cdot)$ as

$$\begin{aligned} \mathcal{K}(w,v) &:= (0, -\frac{1}{\rho a}(\mu v + k w)), \\ \mathcal{B}(r) u &:= (0, \frac{1}{\rho a} b(x;r) u), \\ \mathcal{F}(w,v) &:= (0, -\frac{\alpha}{\rho a} w^3). \end{aligned}$$

The operator $\mathcal{K}$ is a bounded linear operator on $\mathbb{Z}$. For each r, operator $\mathcal{B}(r)$ is also a bounded operator that maps an input $u \in \mathbb{R}$ to the state space $\mathbb{Z}$. Since the space $H^2(0,\ell)$ is contained in the space of continuous functions over $[0,\ell]$, the the nonlinear term w^3 is in $L^2(0,\ell)$. Thus, the nonlinear operator $\mathcal{F}(\cdot)$ is well-defined on $\mathbb{Z}$. Lastly, define the operator $\mathcal{A} = \mathcal{A}_{KV} + \mathcal{K}$, with the same domain as $\mathcal{A}_{KV}$. With these definition and by setting $\mathfrak{z} = (w,v)$, the state space representation of the railway model is described by (1).

The original railway track model in [26] ignores the Kelvin-Voigt damping in the beam (i.e. $C_d = 0$), and only considers the Kelvin-Voigt damping in the ballast. In this case, the semigroup generated by $\mathcal{A}$ is not analytic. In fact, the railway model with $C_d = 0$ constitutes a hyperbolic-like PDE, while the model with $C_d > 0$ is a parabolic-like model. The results of this paper hold true for both models.

To guarantee the existence of a unique solution to the PDE (4), the nonlinear operator $\mathcal{F}(\cdot)$ needs to satisfy Assumptions A2, C1, and C2.

Lemma 4. *The nonlinear operator $\mathcal{F}(\cdot)$ is continuously Fréchet differentiable on $\mathbb{Z}$. This operator is also weakly sequentially continuous on $\mathbb{Z}$.*

The previous lemma also ensures that the nonlinear operator $\mathcal{F}(\cdot)$ is locally Lipschitz continuous on $\mathbb{Z}$ agreeing with Assumption A2. By Theorem 1, for control inputs $u \in L^p_{\text{loc}}(0,\infty)$, $1 < p < \infty$, the existence of a unique local mild solution is guaranteed.

It has been shown that Assumption B and C are satisfied [21]. As a result, the existence of an optimal pair (u^o, r^o) together with an optimal trajectory $\mathfrak{z}^o$ follows from Theorem 2.

Accordingly, using Theorem 3, the optimal pair (u^o, r^o) satisfies equation (3). To derive the equations in (3), some adjoint operators need to be calculated. Calculation of the operator $\mathcal{A}^*$ is straightforward; it is

$$\mathcal{A}^*(f,g) = \left(-g, \frac{1}{\rho a} \mathcal{A}_0 (EIf - C_d g) + \frac{k}{\rho a} f - \frac{\mu}{\rho a} g \right),$$

for all $(f,g) \in D(\mathcal{A}^*)$ where the domain

$$\begin{aligned} D(\mathcal{A}^*) &= \{(f,g) \in \mathbb{Z} |\, g \in H^2(0,\ell) \cap H_0^1(0,\ell), \\ &\qquad EIf - C_d g \in D(\mathcal{A}_0)\}. \end{aligned} \tag{9}$$

Let $\mathfrak{z}^o(t) = (w^o, v^o)$ be the optimal trajectory evaluated at time $t \in [0,T]$. The adjoint of the operator $\mathcal{F}'_{\mathfrak{z}^o(t)}$ for every $t \in [0,T]$ on $\mathbb{Z}$ is

$$\mathcal{F}'^*_{\mathfrak{z}^o(t)}(f,g) = (\zeta, 0). \tag{10}$$

where

$$\begin{aligned} \zeta(x) &= \frac{3\alpha}{\rho a} \int_0^\ell G(x,y)(w^o(y))^2 g(y)\, dy, \\ G(x,y) &= \frac{1}{6\ell} \begin{cases} (2\ell^2 y - 3\ell y^2 + y^3)x + (y-\ell)x^3, & x \leq y \\ (y^3 - \ell^2 y)x + y x^3, & x > y \end{cases} \end{aligned}$$

Furthermore, the adjoint operator of $\mathcal{B}(r)$ as an operator from the input space $\mathbb{R}$ to the state space $\mathbb{Z}$ is derived for every $(f,g) \in \mathbb{Z}$ as

$$\mathcal{B}^*(r)(f,g) = \rho a \int_0^\ell b(x;r) g \, dx. \tag{11}$$

Let $(q_1, q_2) \in \mathbb{Z}$, set $\mathcal{Q}(w,v) = (q_1 w, q_2 v)$ and $\mathcal{R} = 1$ in the cost function of Assumption C5, and let $b_r(x;r^o)$ be the derivative of $b(x;r)$ with respect to r at r^o. In conclusion, the following set of equations yields an optimizer for every initial condition $\mathfrak{z}_0 = (w_0, v_0) \in \mathbb{Z}$:

$$\begin{cases} \rho a w^o_{tt} + (EI w^o_{xx} + C_d w^o_{txx})_{xx} + \mu w^o_t + k w^o \\ +\alpha (w^o)^3 = b(x; r^o) u^o(t), \\ w^o(0,t) = w^o(\ell,t) = 0, \\ EI w^o_{xx}(0,t) + C_d w^o_{txx}(0,t) = 0 \\ EI w^o_{xx}(\ell,t) + C_d w^o_{txx}(\ell,t) = 0 \\ w^o(x,0) = w_0(x),\ w^o_t(x,0) = v_0(x), \end{cases} \tag{12}$$

$$\begin{cases} \rho a f^o_t - \rho a g^o + 3\alpha \int_0^\ell G(x,y)(w^o(y))^2 g^o(y)\, dy \\ \qquad = -\rho a q_1(x) w^o, \\ \rho a g^o_t + (EI f^o_{xx} - C_d g^o_{xx})_{xx} - \mu g^o + k f^o \\ \qquad = -\rho a q_2(x) w^o_t, \\ f^o(0,t) = f^o(\ell,t) = 0,\ g^o(0,t) = g^o(\ell,t) = 0, \\ EI f^o_{xx}(0,t) - C_d g^o_{xx}(0,t) = 0 \\ EI f^o_{xx}(\ell,t) - C_d g^o_{xx}(\ell,t) = 0 \\ f^o(x,T) = 0,\ g^o(x,T) = 0, \end{cases} \tag{13}$$

$$u^o(t) = -\rho a \int_0^\ell b(x; r^o) g^o(x,t)\, dx, \tag{14}$$

$$\int_0^T \int_0^\ell u^o(t) b_r(x; r^o) g^o(x,t)\, dx dt = 0. \tag{15}$$

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