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Extended Abstract: Direct Method to H_∞ Optimal Control and Algorithm Improvements

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Summary: **We address H_∞ control of a class of systems governed by parabolic partial differential equations (PDEs). Closed-form expressions for optimal state feedback with full information are stated. Unlike traditional methods for H_∞-controller synthesis, no iteration is needed to obtain the optimal controller and a closed form expression for the controller is readily calculated. These results are used in improvements to general purpose algorithms for calculating H_∞ control for PDEs and large-scale systems.**

I. INTRODUCTION

The aim of H_∞ control is to stabilize a system and also attenuate its response to disturbances. An iterative approach, based on calculation of the solution to a series of algebraic Riccati equations (AREs), is generally used; see [1] for state feedback and [2] for output feedback of finite-dimensional systems. This approach was extended to infinite-dimensional systems in [3]. In [4], the authors presented a closed-form solution to a H_∞ optimal control problem, for a certain class of systems. This is in contrast to the general purpose method as the ARE needs to be solved iteratively. This has been extended to infinite-dimensional systems in [5].

The direct approach described in [5] is appropriate to H_∞-control of a class of systems governed by partial differential equations (PDEs). Diffusion equations are an important example included in this class. In general, H_∞ synthesis techniques for systems governed by PDEs, particularly those in higher space dimensions, needs to be done by first approximating the given PDE with a finite-dimensional system of possibly large order. Then, controller synthesis requires solving large order AREs in an iterative process to find the optimal attenuation. As for problems originating with finite-dimensional models, the sign-indefiniteness of the quadratic term in the H_∞-ARE, and the need for an iterative procedure to find the optimal attenuation, complicate computation. In [6], the authors address the problem of solution of optimal attenuation for large order systems by extending the game-theoretic algorithm in [7].

In this presentation, we extend the results given in [4] and [5] to formulate the results for finite-dimensional systems on generalized form. A generalized state-space representation is often the result when a PDE is approximated by a finite-dimensional system. This theory is presented in Section II.

Furthermore, we compare the computation time of our closed-form H_∞ laws to synthesis by two general purpose algorithms for a heat equation problem on a two dimensional irregular shape. The direct approach is compared to two general purpose algorithms; the well known Schur algorithm [8] and the algorithm described in [6]. The closed-form approach is several orders of magnitude faster to compute. Finally, in Section III, the theory is used to improve the convergence speed of general purpose algorithms when used on more general classes of equations.

II. OPTIMAL CLOSED-FORM STATE FEEDBACK

In this section, we state the theory concerning closed-form optimal state feedback for a class of parabolic systems. Furthermore, we extend the results given in [4] to systems on generalized state space form. Notice that the first subsection treats infinite-dimensional systems while the second subsection treats finite-dimensional systems.

A. State feedback

Consider a linear time-invariant infinte-dimensional system

$$\frac{dz(t)}{dt} = Az(t) + Bu(t) + Hd(t) \tag{1}$$

where the state $z(t) \in \mathcal{Z}$ and $\mathcal{Z}$ is a Hilbert space. The state $z(t)$ is available for control, and has initial condition $z(0) = 0$. Furthermore, the control signal $u(t) \in \mathcal{U}$ and the disturbance $d(t) \in L_2(0, \infty; \mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces. Furthermore, the operators $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$.

We consider the design of a H_∞-optimal state feedback law to (1) given the following regulated output

$$\zeta(t) = \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}.$$

The transfer function of the closed-loop system from disturbance $d(t)$ to regulated output $\zeta(t)$ with state feedback law $u(t) = Kz(t)$, where $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$, is given by

$$\begin{aligned} \frac{dz(t)}{dt} &= (A + BK)z(t) + Hd(t), \\ \zeta(t) &= \begin{bmatrix} I \\ K \end{bmatrix} z(t). \end{aligned} \tag{2}$$

If K is a stabilizing controller, then $A + BK$ generates an exponentially stable strongly continuous semigroup, i.e., the

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system is stable. Moreover, the Laplace transform of the closed-loop system is denoted G_K, i.e., $\zeta = G_K(s)d$.

The following theorem gives a closed-form expression for a state feedback controller K that minimizes the L_2-gain of G_K, for a certain class of systems (1) namely where the operator A is closed, densely defined, self-adjoint and strictly negative. Note that the notation B^* indicates the adjoint of the operator B. We denote this optimal controller K_{opt}.

Theorem 1 ([5]): Consider the system (1) where A is closed, densely defined, self-adjoint and strictly negative, $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ and $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$, where $\mathcal{Z}$, $\mathcal{U}$ and $\mathcal{V}$ are Hilbert spaces. Then, $\|G_K\|_\infty$ is minimized by the state feedback controller $K_{\text{opt}} = B^* A^{-1}$ and the minimal value of the norm is given by $\|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}}$.

B. Systems with generalized state space representation

Systems with generalized state-space representation, i.e., on descriptor form, are often the result when a PDE is approximated by a finite-dimensional system. Finite-dimensional approximations are often needed in general H_∞ design. Therefore, we treat such systems next. However, we treat a special class of systems on generalized state-space form that are the approximation of the infinite-dimensional systems of the previous subsection.

Let a finite-dimensional system on generalized form be given by

$$\begin{aligned} E\dot{z} &= Az + Bu + Hd, \\ \zeta &= \begin{bmatrix} Lz \\ u \end{bmatrix}, \end{aligned} \tag{3}$$

where $z \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^q$ and matrices E, A, B, H and L are of suitable dimensions. This is also referred to as descriptor form. We consider E symmetric and positive definite, denoted $E \succ 0$ and L to be the Cholesky factorization of E, i.e., $E = L^T L$. Also, we consider A to be symmetric and Hurwitz, i.e., negative definite which we denote $A \prec 0$.

Given a control law $u = Kz$, $K \in \mathbb{R}^{m \times n}$, we denote the transfer function of the closed-loop system by G_K, which is given by

$$G_K(s) = \begin{bmatrix} L \\ K \end{bmatrix} (sE - A - BK)^{-1} H. \tag{4}$$

The following theorem states a closed-form expression for a H_∞ optimal state feedback controller, i.e., a K_{opt} such that the L_2-gain from d to ζ is minimized.

Theorem 2: Consider (4) with $E \succ 0$ and $A \prec 0$. Then, $\|G_K\|_\infty$ is minimized by $K_{\text{opt}} = B^T A^{-1} E$. The minimal value of the norm is given by $\|H^T \left(AE^{-1}A + BB^T\right)^{-1} H\|^{\frac{1}{2}}$.

Remark 1: Notice that the state variable change $\tilde{z} = Lz$ in (3) renders a system on non-generalized state space form with regulated output $\zeta = \begin{bmatrix} \tilde{z}^T & u^T \end{bmatrix}^T$. Thus the vector Lz is related to the continuous state of the PDE, which approximation is given by (3).

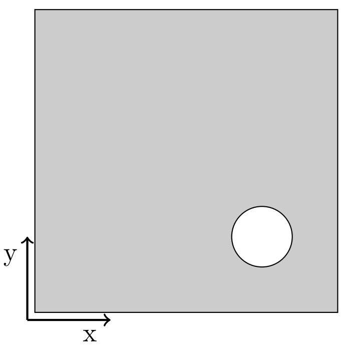

Fig. 1. Irregular geometry considered in algorithm comparison.

III. Algorithm comparison

In this section, we will compare how much faster our method to H_∞ control of parabolic systems is than the computational speed of that of two general purpose algorithms. Our method is simply just to compute the control law in Theorem 2, i.e., the computational time required to compute $B^T A^{-1} E$. The two methods we will compare it with are the Schur algorithm, see [8] for a version for generalized systems, and the method developed in [7] and [6]. The Schur method is provided in MATLAB [9] and thus one of the algorithms readily available for users. It is generally good for small to medium scale problems. The algorithm in [6] is developed for large-scale problems. All simulations are performed using Matlab R2015a.

The following example is based on an example presented in [6]. Consider the heat diffusion problem in two dimensions. The geometry considered is a plane of 4×4 units, with a circle of radius 0.4 units at $(3, 1)$ removed. The lower left corner of the plane is the origin, see Figure 1.

The heat distribution at position (x, y) at time t is denoted $z(x, y, t)$ and governed by

$$\frac{\partial z}{\partial t}(x, y, t) = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + b(x, y)u(t) + h(x, y)d(t), \tag{5}$$

with boundary condition $z(x, y, \cdot) = 0$. Furthermore, the functions $b(x, y)$ and $h(x, y)$ are given by

$$f_{r,\epsilon}(x, y) = \begin{cases} \frac{1}{2\epsilon} & (x, y) \in \{\text{ square centered at } r \\ & \qquad\qquad \text{with side } 2\epsilon\}, \\ 0 & \text{otherwise.} \end{cases}$$

The infinite-dimensional system (5) is approximated by a finite-element method with linear splines. The resulting finite-dimensional system for a given approximation order is on the same form as (3).

We considered $b(x, y) = f_{(2,2),0.5}(x, y)$ and $d(x, y) = 20 \cdot f_{(1,1.3),0.1}(x, y)$ for approximated system order ranging from 20 to 850. The computation time for the Schur method, game theoretic iterative method in [6] and computation of our closed-form solution was compared. Our closed-form solution was at least 100 times faster to compute than that of the other approaches, when used to calculate a controller with fixed attenuation 6, that is $\|G_K\|_\infty < 6$. Notice that the closed-form approach renders an optimal control law. If optimal controllers were to be computed through the general purpose algorithms as well, the gap in computation time would be even larger. The strength in computational speed of our approach could have great implications on lowering the computational time of applications where one needs to solve several H_∞-AREs.

IV. Improvements of general purpose algorithms

In this section, we discuss how insights from the results stated in Section II can be used to increase the convergence rate of general purpose algorithms. That is, methods that can treat systems not only with $E \succ 0$ and $A \prec 0$.

Bisection algorithms for optimal attenuation need a range over which to search for the optimal γ. Generally, this range is set to $[0, \gamma_2]$ where γ_2 is the norm of the closed-loop system given the H_2 controller. The following can be used to specify a tighter range. Given (3) without the assumption on symmetry and Hurwitz stability of A and

$$\zeta = Cz + Du$$

with the assumption that $D^T D = I$, $D^T C = 0$ and $C^T C \succ 0$. Then, the optimal gain is lower bounded as

$$\gamma_{\text{opt}} \geq \|H^T (AE(C^T C)^{-1} E^T A^T + BB^T)^{-1} H\|^{\frac{1}{2}}.$$

This follows from the proof of Theorem 2. Denote the right hand side of the inequality above as γ_{lb}. The narrowed search range can thus be specified as $[\gamma_{\text{lb}}, \gamma_2]$.

Theorem 2 in Section II is applicable to systems (3) with $A \prec 0$ and $E \succ 0$. In closed-loop with the optimal control law, these systems obtain their maximum L_2-gain at frequency zero. In fact, the control law can be derived by only considering the static problem. Then, one only needs to check that the resulting controller can handle the remaining frequencies with L_2-gain smaller than or equal to that at frequency zero. Possibly, the control law given by Theorem 2, more generally $u = B^T A^{-T} Ez$ when there is no assumption on the symmetry of A, can be used for systems with A close to symmetric as well.

V. Conclusions

We present a direct method for H_∞ optimal control to a class of parabolic PDEs. When approximated, the considered PDEs lead to generalized finite-dimensional systems with certain symmetry. We provide the closed-form expression for an H_∞ optimal control law to these finite-dimensional systems. Furthermore, the computational strength of our approach, in terms of the computation time needed, is easily illustrated when compared to general purpose algorithms. Finally, we provide a discussion on how insights made from the theory presented can be used to increase the convergence speed of general purpose algorithms.

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