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Converse Stability Theorems for 2D Systems

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***Abstract*— The paper considers 2D discrete nonlinear systems described by either the Fornasini-Marchesini or Roesser state-space models. Converse stability theorems are developed, which establish that, if an example is exponentially stable, there exists a vector Lyapunov function with particular properties of its entries and the discrete counterpart of divergence. The extension of these results to some class of stochastic models are also given.**

***Key words*: Multidimensional systems, Fornasini-Marchesini model, Roesser model, converse stability theorems, vector Lyapunov functions**

***AMS subject classifications*: 93D05, 93D30, 93E15**

I. INTRODUCTION

Multidimensional (nD) models describe systems with dynamics that evolve in $n > 1$ independent directions. This paper considers 2D discrete nonlinear systems, where the state-space models include the nonlinear versions of the Fornasini-Marchesini [1] and Roesser state-space models [2] and repetitive process models [3] (this reference deals with linear dynamics only). The Roesser model has its origins in image processing problems where the state dynamics are partitioned into two sub-vectors, one for each direction of information propagation and commonly termed the horizontal and vertical respectively. In the Fornasini-Marchesini model [1] a single state vector is used.

Results on the stability of nD nonlinear systems have been reported, e.g., [4]–[10] and the relevant references therein but the area is relatively less well developed than for linear dynamics. Support for the development of a stability and control theory for 2D nonlinear systems that can be extended to stabilization is supplied by examples such as laser metal deposition processes [11], [12]. Moreover, in the linear model case, some results developed for one representation, e.g., linear repetitive processes can also applied to another, e.g., the Roesser model. This is much less likely to be the case for nonlinear dynamics and hence the need to consider the Fornasini-Marchesini and Roesser state-space models separately.

In this paper the interest is in the construction of Lyapunov functions for stable Fornasini-Marchesini and Roesser state-space model descriptions of 2D discrete nonlinear dynamics. In the case of systems that are functions of one indeterminate, also termed standard or 1D in some of the multidimensional systems literature, the importance of converse theorems is that it is possible to prove results on total stability, which is a very important property in applications, see, e.g., [13]–[16] and the references therein.

II. CONVERSE LYAPUNOV THEOREM FOR NONLINEAR FORNASINI-MARCHESINI SYSTEMS

The 2D discrete nonlinear systems considered in this section are described, in the absence of input terms, by the Fornasini-Marchesini state-space model

$$x_{i+1,j+1} = f(x_{i,j+1}, x_{i+1,j}),\ i \geq 0,\ j \geq 0, \qquad (1)$$

where $x_{i,j} \in \mathbb{R}^{n_x}$ is the local state vector and f is a vector-valued function whose entries satisfy $f(0,0) = 0$ and hence there is an equilibrium at the origin. The boundary conditions are assumed to be of the form

$$x_{i,0} = \xi_0(i),\ i \geq 0,\ x_{0,j} = \eta_0(j),\ j \geq 0, \qquad (2)$$

where $\xi_0(i)$ and $\eta_0(j)$ are vectors whose entries are known functions of i and j respectively. Also, it is assumed that there exist finite real numbers $\rho > 0$, $\sigma > 0$ and $0 < \zeta_0 < 1$ such that

$$\begin{aligned} |x_{i,0}|^2 = |\xi_0(i)|^2 \leq \rho \zeta_0^i,\ i \geq 0, \\ |x_{0,j}|^2 = |\eta_0(j)|^2 \leq \sigma \zeta_0^j,\ j \geq 0. \end{aligned} \qquad (3)$$

where throughout this paper $|\cdot|$ denotes the Euclidean norm on vectors.

In this last equation, ζ_0 represents the rate of convergence in i and j of the boundary local state vector sequences. From this point onwards, all references to the boundary conditions will assume that they satisfy (3). In other work, e.g., [4] and the references therein it is assumed that $|\xi_0(i)|$ and $|\eta_0(j)|$ are bounded or constant along i and j on some finite intervals. In this practically motivated case it is routine to obtain the upper bounds in (3). Hence the boundary conditions assumed in this other work are included in (3)

The stability theory developed for 2D discrete linear Fornasini-Marchesini systems has been defined in both the internal, or state, and bounded-input bounded-output settings. Previous research on the stability of 2D discrete nonlinear systems described by this model includes [5], where stability

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and asymptotic stability were defined and sufficient conditions for their existence obtained in a manner similar to the second Lyapunov stability theorem. In [4] sufficient conditions for global asymptotic stability were developed for zero-input 2D digital filters described by a Fornasini-Marchesini state-space model where the nonlinearity arises from a class of overflow errors. This paper further develops the results of [8], [9] towards the construction of Lyapunov functions based on converse Lyapunov theorems. As in the case, of standard, also termed 1D in some of the multidimensional systems, it is to be expected that this stronger form of stability will be required in at least some applications, the starting stability definition is given next.

Definition 1: A 2D discrete nonlinear 2D system described by the Fornasini-Marchesini state-space model (1) is said to be exponentially stable if for all boundary conditions satisfying (3) there exist $\kappa > 0$ and $0 < \lambda < 1$ such that

$$|x_{i,j}|^2 \leq \kappa \lambda^{i+j},\ i,\ j \geq 0. \tag{4}$$

Exponential stability for the linear dynamics case was considered in [17]. This stability theory is physically motivated by applications where such a system would never be operated with boundary conditions that can diverge as the dynamics evolve. For the autonomous case under consideration the state vector must decay to zero as $i+j \to \infty$, which results in a strong form of stability since the boundary conditions are required to have a uniform convergence property.

One way of characterizing exponential stability would be to attempt to use a Lyapunov function approach as in the stability analysis of 1D nonlinear systems. The Lyapunov approach is based on properties of the function itself and for discrete dynamics of its increments, but the dynamics considered are determined by vector-valued functions of the two independent variables i and j. A candidate Lyapunov function for these systems could be chosen as a scalar function, say $\tilde{V}(i,j)$, but to construct the gradient along the system trajectories it is required to have $x_{i+1,j} - x_{i,j}$ and $x_{i,j+1} - x_{i,j}$. These quantities can only be found by solving (1), but then all of the advantages of the Lyapunov approach are lost.

As an alternative, previous work has used a vector Lyapunov functions approach for other classes of 2D nonlinear systems, see, e.g., [7], where for discrete dynamics a counterpart of divergence was used instead of the gradient. The analysis that follows uses a similar setting to characterize the property of Definition 1 using a vector Lyapunov function of the form

$$V(x_{i,j+1}, x_{i+1,j}) = \begin{bmatrix} V_1(x_{i,j+1}) \\ V_2(x_{i+1,j}) \end{bmatrix}, \tag{5}$$

where $V_1(x) > 0,\ x \neq 0,\ V_2(x) > 0,\ x \neq 0,\ V_1(0) = 0, V_2(0) = 0$. Also the counterpart of the divergence operator of this function along the trajectories of (1) is

$$\begin{aligned} \mathscr{D}V(x_{i,j+1}, x_{i+1,j}) &= V_1(x_{i+1,j+1}) - V_1(x_{i,j+1}) \\ &\quad + V_2(x_{i+1,j+1}) - V_2(x_{i+1,j}). \end{aligned} \tag{6}$$

The following is a converse theorem for this stability property.

Theorem 1: Suppose that a 2D discrete nonlinear system described by (1) with boundary conditions satisfying (3) is exponentially stable in the sense of Definition 1. Then there exists a vector Lyapunov function (5) with entries that are bounded on the solutions of (1) and satisfies, for some positive constants c_1, and c_3 the following inequalities

$$V_1(x_{i,j}) \geq c_1 |x_{i,j}|^2, \tag{7}$$
$$V_2(x_{i,j}) \geq c_1 |x_{i,j}|^2, \tag{8}$$
$$\mathscr{D}V(x_{i,j+1}, x_{i+1,j}) \leq -c_3(|x_{i,j+1}|^2 + |x_{i+1,j}|^2). \tag{9}$$

Proof: Let $x_{r,s},\ r \geq 0,\ s \geq 0$ be a exponentially stable solution of (1) with boundary conditions satisfying (2). Define on this solution the entries in the function $V(x)$ of (5) such that

$$V_1(x_{i,j}) = \alpha \sum_{m=i}^{\infty} \sum_{q=j}^{\infty} |x_{m,q}|^2,$$
$$V_2(x_{i,j}) = \beta \sum_{m=i}^{\infty} \sum_{q=j}^{\infty} |x_{m,q}|^2,\ \alpha > 0,\ \beta > 0.$$

By exponential stability V_1 and V_2 are well defined. Moreover, these functions are uniquely defined since the solutions of (1) are uniquely defined by the boundary conditions assumed. Choosing $c_1 \leq \min(\alpha, \beta)$ gives

$$c_1 |x_{i,j}|^2 \leq \alpha \sum_{m=i}^{\infty} \sum_{q=j}^{\infty} |x_{m,q}|^2 = V_1(x_{i,j}),$$
$$c_1 |x_{i,j}|^2 \leq \beta \sum_{m=i}^{\infty} \sum_{q=j}^{\infty} |x_{m,q}|^2 = V_2(x_{i,j})$$

and both (7) and (8) are valid. By the exponential stability property

$$\sum_{m=i}^{\infty} \sum_{q=j}^{\infty} |x_{m,q}|^2 \leq \kappa \sum_{m=i}^{\infty} \sum_{q=j}^{\infty} \lambda^m \lambda^q = \frac{\kappa}{(1-\lambda)^2}$$

and it follows that V_1 and V_2 are bounded on the solutions of (1) with the required properties. Moreover,

$$\begin{aligned} \mathscr{D}V(x_{i,j+1}, x_{i+1,j}) &= V_1(x_{i+1,j+1}) - V_1(x_{i,j+1}) \\ &\quad + V_2(x_{i+1,j+1}) - V_2(x_{i+1,j}) \\ &= \alpha \left(\sum_{m=i+1}^{\infty} \sum_{q=j+1}^{\infty} |x_{m,q}|^2 - \sum_{m=i}^{\infty} \sum_{q=j+1}^{\infty} |x_{m,q}|^2 \right) \\ &\quad + \beta \left(\sum_{m=i+1}^{\infty} \sum_{q=j+1}^{\infty} |x_{m,q}|^2 - \sum_{m=i+1}^{\infty} \sum_{q=j}^{\infty} |x_{m,q}|^2 \right) \\ &= -\alpha \sum_{q=j+1}^{\infty} |x_{i,q}|^2 - \beta \sum_{m=i+1}^{\infty} |x_{m,j}|^2 \\ &\leq -\alpha |x_{i,j+1}|^2 - \beta |x_{i+1,j}|^2 \\ &\leq -\min(\alpha, \beta)(|x_{i,j+1}|^2 + |x_{i+1,j}|^2) \end{aligned}$$

and it follows that (9) holds with $c_3 = \min(\alpha, \beta)$. This completes the proof. ■

III. CONVERSE LYAPUNOV THEOREM FOR NONLINEAR ROESSER SYSTEMS

In the linear dynamics case it is often possible to transfer results easily between Fornasini-Marchesini and Roesser models and vice versa. This is not the case for nonlinear dynamics and this section considers 2D discrete nonlinear systems described, in the absence of input terms, by the Roesser state-space model

$$\begin{aligned} x^h_{i+1,j} &= f_1(x^h_{i,j}, x^v_{i,j}), \\ x^v_{i,j+1} &= f_2(x^h_{i,j}, x^v_{i,j}),\ i \geq 0,\ j \geq 0, \end{aligned} \tag{10}$$

where $x^h \in \mathbb{R}^{n_h}$ is horizontal state vector, $x^v \in \mathbb{R}^{n_v}$, is vertical state vector, f_1, and f_2 are vector-valued functions, where it is assumed that the entries are such that $f_1(0,0) = 0$, $f_2(0,0) = 0$ and hence an equilibrium at the origin. The boundary conditions are assumed to be of the form

$$\begin{aligned} x^h_{0,j} &= \xi^h_0(j),\ j \geq 0, \\ x^v_{i,0} &= \xi^v_0(i),\ i \geq 0, \end{aligned} \tag{11}$$

where $\xi^h_q(i)$ and $\xi^h_m(j)$ are vectors whose entries are known functions of i and j respectively. It is assumed there exist finite real numbers $\alpha_h > 0$, $\alpha_v > 0$ and $0 < \zeta_0 < 1$ such that

$$\begin{aligned} |x^h_{0,j}|^2 &= |\xi^h_0(j)|^2 \leq \alpha_h \zeta^j_0,\ j \geq 0, \\ |x^v_{i,0}|^2 &= |\xi^v_0(i)|^2 \leq \alpha_v \zeta^i_0,\ i \geq 0. \end{aligned} \tag{12}$$

From this point onwards, all references to the boundary conditions will assume that they satisfy (12). These boundary conditions include the important practically motivated case when $|\xi^h_0(j)|$ and $|\xi^v_0(i)|$ are bounded or constant along i and j over finite intervals.

Introduce the full state vector as $x_{i,j} = [x^{hT}_{i,j}\ x^{vT}_{i,j}]^T$. Then the following is the definition of exponential stability for the model considered.

Definition 2: A 2D discrete nonlinear system described by (10) is said to be exponentially stable if for all boundary conditions satisfying (12) there exist $\kappa > 0$ and $0 < \lambda < 1$, such that

$$|x_{i,j}|^2 \leq \kappa \lambda^{i+j},\ i \geq 0,\ j \geq 0. \tag{13}$$

Remark 1: An alternative definition of exponential stability for 2D discrete nonlinear systems described the Roesser model is given in [6].

Consider a vector Lyapunov function of the form

$$V(x_{i,j}) = \begin{bmatrix} V_1(x^h_{i,j}) \\ V_2(x^v_{i,j}) \end{bmatrix}, \tag{14}$$

where $V_1(x) > 0,\ x \neq 0,\ V_2(x) > 0,\ x \neq 0,\ V_1(0) = 0, V_2(0) = 0$. The counterpart of the divergence operator of this function along the trajectories of (10) is

$$\mathscr{D}V(x_{i,j}) = V_1(x^h_{i+1,j}) - V_1(x^h_{i,j}) + V_2(x^v_{i,j+1}) - V_2(x^v_{i,j}). \tag{15}$$

The following converse theorem can now be established.

Theorem 2: Suppose that a discrete nonlinear 2D system described by (10) and (11) is exponentially stable. Then there exists a vector Lyapunov function (14) whose entries are bounded on the solutions to (10) and satisfies for some positive constants c_1 and c_2, the inequalities

$$\begin{aligned} V_1(x^h_{i,j}) &\geq c_1|x^h_{i,j}|^2, && (16) \\ V_2(x^v_{i,j}) &\geq c_2|x^v_{i,j}|^2, && (17) \\ \mathscr{D}V(x_{i,j}) &\leq -c_3(|x^h_{i,j}|^2 + |x^v_{i,j}|^2). && (18) \end{aligned}$$

Proof: Define on the solutions to (10) with boundary conditions (11) the entries in the function (14) as

$$V_1(x^h_{i,j}) = \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} |x^h_{m,q}|^2,\ V_2(x^v_{i,j}) = \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} |x^v_{m,q}|^2.$$

By exponential stability, V_1 and V_2 are well defined. Also these functions are uniquely defined because the solutions of (10) are uniquely defined by the boundary conditions assumed. Choosing $c_1 \leq 1$ gives

$$\begin{aligned} c_1|x^h_{i,j}|^2 &\leq \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} |x^h_{m,q}|^2 = V_1(x^h_{i,j}), \\ c_2|x^v_{i,j}|^2 &\leq \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} |x^v_{m,q}|^2 = V_2(x^v_{i,j}). \end{aligned}$$

Also by exponential stability

$$\begin{aligned} \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} |x^h_{m,q}|^2 &\leq \kappa \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} \lambda^{m-i}\lambda^{q-j} = \frac{\kappa}{(1-\lambda)^2}, \\ \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} |x^v_{m,q}|^2 &\leq \kappa \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} \lambda^{m-i}\lambda^{q-j} = \frac{\kappa}{(1-\lambda)^2}. \end{aligned}$$

and it follows that both V_1 and V_2 are bounded along the trajectories of (10). Moreover,

$$\begin{aligned} \mathscr{D}V(x_{i,j}) &= V_1(x^h_{i+1,j}) - V_1(x^h_{i,j}) + V_2(x^v_{i,j+1}) - V_2(x^v_{i,j}) \\ &= \left(\sum_{m=i+1}^{\infty}\sum_{q=j}^{\infty} |x^h_{mq}|^2 - \sum_{m=i}^{\infty}\sum_{q=j}^{\infty} |x^h_{m,q}|^2 \right) \\ &+ \left(\sum_{m=i}^{\infty}\sum_{q=j+1}^{\infty} |x^v_{m,q}|^2 - \sum_{m=i=}^{\infty}\sum_{q=j}^{\infty} |x^v_{m,q}|^2 \right) \\ &= -\sum_{q=j}^{\infty} |x^h_{i,q}|^2 - \sum_{m=i}^{\infty} |x^v_{m,j}|^2 \leq -|x^h_{i,j}|^2 - |x^v_{i,j}|^2 \end{aligned}$$

and it follows immediately that (18) holds with $c_3 \leq 1$ and the proof is complete. ■

IV. STOCHASTIC FORNASINI-MARCHESINI SYSTEMS

This section considers stochastic 2D discrete nonlinear systems described by the Fornasini-Marchesini state-space model

$$\begin{aligned} x_{i+1,j+1} &= f(x_{i,j+1}, x_{i+1,j}) + G_1(x_{i,j+1}, x_{i+1,j})v_{i,j+1} \\ &+ G_2(x_{i,j+1}, x_{i+1,j})v_{i+1,j},\ i \geq 0,\ j \geq 0, \end{aligned} \tag{19}$$

where $x_{i,j} \in \mathbb{R}^{n_x}$ is the local state vector, $v_{i,j}$ is a vector-valued discrete zero mean random process such that $\mathrm{E}[v_{i,k}v^T_{j,l}] = 0$ if $i \neq j$ or $k \neq l$, $\mathrm{E}[v_{i,j}v^T_{i,j}] = I$, E denotes the expectation operator, f is a vector-valued function such that $f(0,0) = 0$, G_i, $i = 1,2$, are matrix-valued functions, such that $G_i(0,0) = 0$, $i = 1,2$ and hence an equilibrium at the origin. The boundary conditions are assumed to be of the following form (the stochastic counterpart of (2))

$$\begin{aligned} \mathrm{E}[|x_{i,0}|^2] &= \mathrm{E}[|\xi_0(i)|^2] \leq \rho\zeta_0^i,\ i \geq 0, \\ \mathrm{E}[|x_{0,j}|^2] &= \mathrm{E}[|\eta_0(j)|^2] \leq \sigma\zeta_0^j,\ j \geq 0. \end{aligned} \tag{20}$$

where $\xi_0(i)$ and $\eta_0(j)$ are vectors whose entries are known random sequences on $i \geq 0$ and $j \geq 0$, respectively, and $0 < \zeta_0 < 1$. In particular, ζ_0 represents the rate of convergence in $i \geq 0$ and $j \geq 0$ of the mean-square norms of the boundary local state vector sequences. From this point onwards, all references to the boundary conditions will assume that they satisfy (20). It is also assumed that $v_{i,j}$ does not depend on the sequences $\xi_0(i)$ and $\eta_0(j)$ for all i, j. As in deterministic case the boundary conditions (20) include the case when $\mathrm{E}[|\xi_0(i)|^2]$ and $\mathrm{E}[|\eta_0(j)|^2]$ are bounded or constant along i and j on some finite intervals.

Definition 3: A 2D discrete nonlinear system described by (19) is said to be exponentially stable in the mean square if for all boundary conditions satisfying (20) there exist $\kappa > 0$ and $0 < \lambda < 1$, such that

$$\mathrm{E}[|x_{i,j}|^2] \leq \kappa\lambda^{i+j}. \tag{21}$$

The analysis that follows makes use of a vector function of the form (5) and the stochastic counterpart of the divergence operator of this function along the trajectories of (19) is

$$\begin{aligned} \mathscr{D}_{\xi,\eta}V(x_{i,j+1},x_{i+1,j}) &= \mathrm{E}[V_1(x_{i+1,j+1}) - V_1(x_{i,j+1}) \\ &+ V_2(x_{i+1,j+1}) - V_2(x_{i+1,j})|x_{i,j+1} = \xi,\ x_{i+1,j} = \eta]. \end{aligned} \tag{22}$$

The following converse theorem can now be established.

Theorem 3: Suppose that a 2D discrete nonlinear system described by (19) and (20) is exponentially stable in the mean square in the sense of Definition 3. Then there exists a vector function of the form (5), whose entries are bounded in the mean square along the trajectories (19) and satisfy for some positive constants c_1 and c_3 and the inequalities

$$V_1(\xi_{i,j}) \geq c_1|\xi_{i,j}|^2, \tag{23}$$

$$V_2(\eta_{i,j}) \geq c_1|\eta_{i,j}|^2, \tag{24}$$

$$\mathscr{D}_{\xi,\eta}V(x_{i,j+1},x_{i+1,j}) \leq -c_3(|\xi|^2 + |\eta|^2). \tag{25}$$

Proof: Define on the solutions of (19) with boundary conditions (20) the entries in the vector function (5) in the form

$$\begin{aligned} V_1(\xi_{i,j}) &= \alpha\sum_{m=i}^{\infty}\sum_{q=j}^{\infty}\mathrm{E}[|x_{m,q}|^2|x_{i,j} = \xi_{i,j}], \\ V_2(\eta_{i,j}) &= \beta\sum_{m=i}^{\infty}\sum_{q=j}^{\infty}\mathrm{E}[|x_{m,q}|^2|x_{i,j} = \eta_{i,j}]\ \alpha > 0,\ \beta > 0. \end{aligned}$$

Since exponential stability in the mean square holds, both V_1 and V_2 are well defined. Also choosing $c_1 \leq \min(\alpha, \beta)$ gives

$$c_1|\xi_{i,j}|^2 \leq \alpha\sum_{m=i}^{\infty}\sum_{q=j}^{\infty}\mathrm{E}[|x_{m,q}|^2|x_{i,j} = \xi_{i,j}] = V_1(\xi_{i,j}),$$

$$c_1|\eta_{i,j}|^2 \leq \beta\sum_{m=i}^{\infty}\sum_{q=j}^{\infty}\mathrm{E}[|x_{m,q}|^2|x_{i,j} = \eta_{i,j}] = V_2(\eta_{i,j})$$

and it follows from the condition for exponential stability that

$$\sum_{m=i}^{\infty}\sum_{q=j}^{\infty}\mathrm{E}[|x_{m,q}|^2] \leq \kappa|x_{i,j}|^2\sum_{m=i}^{\infty}\sum_{q=j}^{\infty}\lambda^{m-i}\lambda^{q-j} = \frac{\kappa}{(1-\lambda)^2}.$$

Choosing $c_2 \geq \max(\alpha, \beta)\frac{\kappa}{(1-\lambda)^2}$ gives that (23) and (24) hold. Also, using properties of conditional expectation gives

$$\begin{aligned} &\mathscr{D}_{\xi,\eta}V(x_{i,j+1},x_{i+1,j}) = \mathrm{E}[V_1(x_{i+1,j+1}) - V_1(x_{i,j+1}) \\ &\quad + V_2(x_{i+1,j+1}) - V_2(x_{i+1,j})|x_{i,j+1} = \xi,\ x_{i+1,j} = \eta] \\ &= \alpha\mathrm{E}\left[\sum_{m=i+1}^{\infty}\sum_{q=j+1}^{\infty}|x_{m,q}|^2 - \sum_{i=m}^{\infty}\sum_{j=q+1}^{\infty}|x_{m,q}|^2\right| x_{i,j+1} = \xi, \\ &x_{i+1,j} = \eta] + \beta\mathrm{E}\left[\sum_{m=i+1}^{\infty}\sum_{q=j+1}^{\infty}|x_{m,q}|^2 - \sum_{m=i+1}^{\infty}\sum_{q=j}^{\infty}|x_{m,q}|^2\right| x_{i,j+1} \\ &= \xi, x_{i+1,j} = \eta] = \mathrm{E}\left[-\alpha\sum_{q=j+1}^{\infty}|x_{m,q}|^2 - \beta\sum_{m=i+1}^{\infty}|x_{m,q}|^2\right| \\ &\quad x_{i,j+1} = \xi,\ x_{i+1,j} = \eta] \leq -\alpha|\xi|^2 - \beta|\eta|^2 \\ &\quad \leq -\min(\alpha,\beta)(|\xi|^2 + |\eta|^2). \end{aligned}$$

It follows immediately that (25) holds with $c_3 = \min(\alpha, \beta)$ and the proof is complete. ■

V. CONCLUSIONS

The results developed in this paper generalize to 2D discrete nonlinear systems described by the Fornasini-Marchesini and Roesser state-space models the fact that it is possible to construct a Lyapunov function only if solution of the considered systems is known. In case of linear 1D systems, the Lyapunov function is in homogeneous form and application of the converse theorem gives necessary and sufficient stability conditions, which easily are reduced to the algebraic problem of solvability of the matrix Lyapunov equation/inequality. The extension of this result to the systems considered in this paper is one area to which future research could be directed.

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