Symbolic computation and hyperbolic polynomials*

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***Abstract*— Hyperbolic polynomials are multivariate generalizations of characteristic polynomials of real symmetric matrices. Hyperbolic programming is the problem of computing the infimum of a linear function when restricted to the hyperbolicity cone of a hyperbolic polynomial, a generalization of semidefinite programming. In the talk I will discuss an approach based on symbolic computation, relying on the multiplicity structure of the algebraic boundary of the cone, without the assumption of determinantal representability of the given polynomial. This allows us to design exact algorithms able to certify the multiplicity of the solution and the optimal value of the linear function. This is joint work with Daniel Plaumann, University of Dortmund.**

In the rest of the talk, I will discuss a work in progress, joint with Rainer Sinn, FU Berlin.

Semidefinite programming (SDP) represents a very important class of convex optimization problems for which approximate solutions can be computed through a variety of numerical algorithms, the most efficient of which is primal-dual interior point methods. Recently, exact algorithms for semidefinite programming have been developed in [9]. The strategy is based in constructing lifting of low-rank matrix varieties.

While symbolic algorithms obviously have a much higher complexity than numerical ones, finding exact solutions has many benefits, especially regarding certification of information about the solution. The important information for SDP are the rank of the optimizer, that can be determined exactly by the algorithm in [9]. In the work [15], an algorithm for solving SDP programs with rank constrained has been obtained: this is a non-convex optimization problem that models, for instance, the computation of sums-of-squares certificates of low length. In this paper, joint with D. Plaumann, we consider analogous algorithmic questions in the more general framework of *hyperbolic programming*.

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A real homogeneous polynomial f in $x = (x_1, \ldots, x_n)$ is called *hyperbolic* with respect to a direction $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and if $f(te - a) \in \mathbb{R}[t]$ has only real roots for every $a \in \mathbb{R}^n$. The general determinant of symmetric matrices has this property with respect to the unit matrix $e = \mathbb{I}_d$, since $\det(t\mathbb{I}_d - A)$ is the classical characteristic polynomial of the real symmetric matrix A.

Hyperbolic polynomials can therefore be seen as generalized characteristic polynomials. If f is hyperbolic with respect to e, the hypersurface defined by f bounds a convex cone containing e, the *hyperbolicity cone*, generalizing the cone of positive semidefinite matrices in case of the determinant. The zeros of $f(te - a)$ can be regarded as generalized eigenvalues of $a \in \mathbb{R}^n$, and the multiplicity of the root $t = 0$ of $f(te - a)$ as the corank of a. A *hyperbolic program* is the convex optimization problem asking to minimize a linear function over the hyperbolicity cone of a hyperbolic polynomial.

Such cones have non-empty interior by construction (the interior will indeed contain the point e). Denote by $S_d(\mathbb{R})$ the set of $d \times d$ real symmetric matrices. *Regular* semidefinite programs (in which the feasible set has non-empty interior) correspond to the case in which f is the restriction of the determinant map $\det\colon S_d(\mathbb{R}) \to \mathbb{R}$ to a linear subspace $V \subset S_d(\mathbb{R})$ containing a positive definite matrix. More precisely, for such V, the polynomial $f = \det\big|_V$ is hyperbolic, and its hyperbolicity cone is the spectrahedron $V \cap S_d^+(\mathbb{R}) = \{M \in V : M \succeq 0\}$. If V does not contain positive definite matrices, $V \cap S_d^+(\mathbb{R})$ is still a spectrahedron, but $f = \det\big|_V$ is not hyperbolic.

Moreover, not every hyperbolic polynomial can be represented in this way (in fact, the set of representable polynomials is, in general, of strictly smaller dimension) which motivates the development of techniques that are independent of the determinantal representability of f. Hyperbolic programming can be solved numerically with interior point methods much like SDP [4], [8], [16].

One of the major challenges in hyperbolic programming, when compared to SDP, is the lack of an explicit duality theory, while SDP duality is always heavily exploited. The methods in [9] rely on the good properties of determinantal varieties, which provide an explicit non-singular lifting of the variety of symmetric matrices of bounded rank in a given subspace. The same is not available for hyperbolic programming. However, hyperbolicity of a real polynomial still imposes some strong conditions on the structure of the real part of the singular locus of the hypersurface.

Given a polynomial f, hyperbolic with respect to $e \in \mathbb{R}^n$, let Λ_+ be the hyperbolicity cone of f, and let $\Gamma_m \subset \mathbb{R}^n$ denote the set of points of multiplicity at least m. Furthermore, let $L_e = \{x \in \mathbb{R}^n : e^T x = 1\}$ be the affine space orthogonal to the direction e (containing $\frac{e}{\|e\|^2}$) and write $\Lambda'_+ = \Lambda_+ \cap L_e$ and $\Gamma'_m = \Gamma_m \cap L_e$. We show that if m equals the maximal multiplicity on Λ'_+, then Λ'_+ contains one of the real connected components of Γ'_m, proving that $\Gamma'_m \cap \Lambda_+$ is the union of some components of Γ'_m. Thus a point of maximal multiplicity (analogous to the minimal rank in SDP) can be found by sampling the connected components of Γ'_m. Since this is an algebraic set (rather than just semialgebraic), this reduces to a standard problem in computational real algebraic geometry. Furthermore, we show that the more general convex hyperbolic programming problem over Λ'_+ is equivalent to computing local minimizers over the sets Γ'_m of the same linear function. This can be carried out in practice using Lagrange multipliers, provided that the corresponding set of critical points has complex dimension 0. We use these results to design an exact algorithm for hyperbolic programming. Applying this to explicit examples yields interesting results that will be discussed during the talk.

In the rest of the talk, I will describe a recent work in progress with Rainer Sinn, FU Berlin. When passing from linear to semidefinite and hyperbolic programming, many nice properties of the optimization problem are preserved. By the way, the nonlinearity can often introduce degenerate properties such as weak infeasibility. In our joint project, we deal with the problem of characterizing weak infeasibility in the case of hyperbolic programming and of more general conic programs.

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