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A finite difference scheme for iterative learning control oriented modelling of physical systems described by PDEs in polar coordinates

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Abstract— **Iterative learning control is a well established area of research and applications in engineering and more recently healthcare. This form of control is especially suitable for applications where the system is required to make repeated executions of a finite duration task with resetting to the starting location at the end of each execution. The majority of the theory and designs are for systems described by finite dimensional differential or discrete dynamics but there is increasing interest in the extension to systems described by partial differential equations, where the interest in this paper is in the use of approximate finite dimensional models of the dynamics. The construction of such models studied using a regular hexagonal grid and polar coordinates, with a fourth order partial differential equation as an example. A supporting numerical application study is also given.**

I. INTRODUCTION

Systems with spatial and temporal dynamics described by partial differential equations (PDEs) arise in many applications. Moreover, the design and physical implementation of control laws for such systems will require discretisation of the defining PDEs. Form the numerical literatures, one group of methods that can be applied is a finite difference approximation [1]. A critical factor with this general approach is numerical stability (convergence), i.e., the trajectories produced by the discrete approximation must be close, as measured by some appropriate measure, to those produced by the original PDE. One method of checking for checking numerical stability is due to von Neumann, see,e.g., [2].

Discretization of PDEs with one temporal and one spatial variables, e.g., the heat transfer equation, results in models that are structurally very similar to repetitive processes [3]. The these process make a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration termed the pass length. The sequence of operation is that a pass is completed and then the process is reset to the starting location and the next one can begin, either immediately after the resetting is complete or after a further period of time has elapsed. On each pass, an output, termed the pass profile, is produced which acts as a forcing function on and therefore contributes to the dynamics of the next pass profile.

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These processes are therefore a particular case of a 2D system where there are two independent directions of the information propagation. In the repetitive process representation of the discretization of PDEs, the pass number is associated with the discrete time sample instants and the along pass dynamics are governed by the discrete spatial variable, see, e.g., [4]. The use of this setting to design iterative learning control laws for spatio-temporal dynamics has been reported, see, e.g., [4].

Explicit methods are one class of finite difference discretization schemes [5], [6] and one of these was used in [4]. These methods produce a causal in time discrete recursive model where, in the repetitive process interpretation, at any instant on the current pass a window of sample instants on the previous pass contribute to the dynamics. However, explicit discretization methods are conditionally numerically stable, i.e., the time discretization period is related to its spatial counterpart. This, in effect, leads to the need to use dense time and spatial discretization grids.

One way of overcoming this drawback is to use the so-called singular methods, see [5], [6], [7] and, particular, the *Crank-Nicolson* method [8] that can produce an unconditionally stable discrete approximation to the dynamics of the original PDE. Hence, the time and spatial grids are not related and can therefore be less dense. However, the resulting discrete models are in implicit form, i.e., there is no straightforward dependence of the pass profile at any instance on the current pass and the window of previous pass values. Instead, this dependence is between windows of sample instant data generate on the current and previous passes, see [9].

Formulating and solving control problems for singular systems requires the use of the lifting approach, i.e., absorbing the spatial structure of the system into possibly high dimensional vectors, again see [9], [10] for a detailed treatment. In [10] the *Crank-Nicolson* method was extended to systems described by a PDE defined over time and two spatial variables. As a particular example, a thin circular flexible plate was considered, which, e.g., can be used to model the vibrations of a deformable mirror subject to a transverse external force. In this paper a regular hexagonal grid with polar coordinates is used for discretization where Cartesian coordinates were used. It is shown that the resulting discrete approximation has the unconditional numerical stability property and hence, relative to the discrete approximations discussed above, a significantly less dense discretization grid can be used with no degradation of the approximate model dynamics. This, in turn, means a much smaller number of sensors and actuators distributed over a controlled plate can (potentially) be used to advantage in control law design and implementation.

II. DESCRIPTION OF A CIRCULAR DEFORMABLE MIRROR IN POLAR COORDINATES

A circular deformable mirror can be described by partial differential equation [11, p. 283, eq. (191)]

$$\frac{1}{r^3}\frac{\partial w}{\partial r}+\frac{4}{r^4}\frac{\partial^2 w}{\partial \varphi^2}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}-\frac{2}{r^3}\frac{\partial^3 w}{\partial r\,\partial \varphi^2}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}+\frac{1}{r^4}\frac{\partial^4 w}{\partial \varphi^4} + \frac{2}{r^2}\frac{\partial^4 w}{\partial r^2\,\partial \varphi^2}+\frac{\partial^4 w}{\partial r^4}+\frac{\rho\, h}{D}\frac{\partial^2 w}{\partial t^2}=\frac{f}{D}, \quad (1)$$

where $w = w(t, r, \varphi)$ denotes the lateral deflection (m) of the mirror at time t (s) in polar coordinates r (radius) (m) and φ (angle) (rad). Also ρ denotes the mass per unit area ($\mathrm{kg\,m^{-2}}$) and h is the thickness of the plate (m), $D = \frac{E\,h^3}{12(1-\nu^2)}$, where ν is the Poisson ratio, E is Young's modulus [$\mathrm{N\,m^{-2}}$] and f is the transverse external force with dimension of force per unit area ($\mathrm{N\,m^{-2}}$). In the case when the load is symmetrically distributed with respect to the center of the mirror, the deflection is independent of φ and (1) becomes, see again [11],

$$\frac{1}{r^3}\frac{\partial w}{\partial r}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}+\frac{\partial^4 w}{\partial r^4}+\frac{\rho\, h}{D}\frac{\partial^2 w}{\partial t^2}=\frac{f}{D}. \quad (2)$$

In this paper, the load is taken as symmetrically distributed with respect to the center of the mirror and described by (2). Further, the mirror diameter is equal to a and it is assumed that the edges of the mirror are clamped. Hence, the deflection and its first derivation with respect to r are zero at all points, where $r = a/2$, i.e.,

$$w(t, a/2) = 0, \qquad \frac{\partial}{\partial r}w(t, a/2) = 0 \quad (3)$$

for all $t \geq 0$.

The mirror is assumed to be equipped with an array of sensors and actuators that are arranged in a hexagonal grid and is shown in Fig. 1. In this figures the locations with sensors and actuators are shown in black. The meaning to the other points, denoted by red and blue colors, will be detailed later in the paper. The number of nodes placed on the diameter is denoted by n ($n = 5$ in the example shown in Fig. 1. As one possible application, consider a tracking example where the mirror is required to have a particular shape after a period time. Then as discussed in [9], iterative learning control can be applied. This area requires further research on both the construction of the approximate model used for control design and the control law to be applied and is discussed again in the last section of this paper.

III. DISCRETISATION

The control of the deformable mirror in this paper is based on signals applied by the array of actuators with the resulting deflections measured by the array of sensors the PDE (2) will be discretized in the spatial variable to obtain a model on which to base controller design. The control implementation is in discrete-time and therefore discretiation is also implemented for the temporal variable.

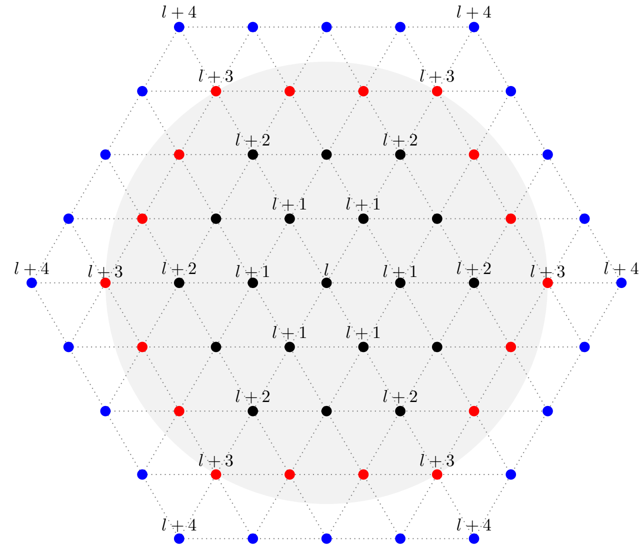

Fig. 1. An example of the hexagonal grid for $n = 5$

Discretisation is based on finite difference methods, where basic principle is to cover the region where a solution is sought by a regular grid and to replace derivatives by differences using only values at these nodal points. In the case considered, these nodal points are specified by the location of sensors and actuators and are given in Fig. 1. Let p and l denote, respectively, t_p and r_l and δ_t and δ_r denote, respectively, the sampling period and the distance between nodes and

$$\delta_r = \frac{a}{n+1}. \quad (4)$$

Also r, $0 \leq r \leq a/2$, denotes distance from the middle of the plate. In what follows the case when $r > 0$ and $r = 0$ are considered.

A. Case $r > 0$

In this case, the second derivative with respect to time is approximated by

$$\frac{\partial^2 w(t,r)}{\partial t^2} \approx \frac{1}{\delta_t^2}\left(w_{p+2,l} - 2\,w_{p+1,l} + w_{p,l}\right). \quad (5)$$

and the other derivatives are approximated by

$$\frac{\partial w(t,r)}{\partial r} \approx \frac{1}{6\,\delta_r}(w_{p+2,l+1} - w_{p+2,l-1} + w_{p+1,l+1} - w_{p+1,l-1} + w_{p,l+1} - w_{p,l-1}), \quad (6)$$

$$\frac{\partial^2 w(t,r)}{\partial r^2} \approx \frac{1}{3\,\delta_r^2}(w_{p+2,l+1} - 2\,w_{p+2,l} + w_{p+2,l-1} + w_{p+1,l+1} - 2\,w_{p+1,l} + w_{p+1,l-1} + w_{p,l+1} - 2\,w_{p,l} + w_{p,l-1}), \quad (7)$$

$$\frac{\partial^3 w(t,r)}{\partial r^3} \approx \frac{1}{6\,\delta_r^3}\big(w_{p+2,l+2} - 2\,w_{p+2,l+1} + 2\,w_{p+2,l-1} - w_{p+2,l-2} + w_{p+1,l+2} - 2\,w_{p+1,l+1} + 2\,w_{p+1,l-1} - w_{p+1,l-2} + w_{p,l+2} - 2\,w_{p,l+1} + 2\,w_{p,l-1} - w_{p,l-2}\big), \tag{8}$$

$$\frac{\partial^4 w(t,r)}{\partial r^4} \approx \frac{1}{3\,\delta_r^4}\big(w_{p+2,l+2} - 4\,w_{p+2,l+1} + 6\,w_{p+2,l} - 4\,w_{p+2,l-1} + w_{p+2,l-2} + w_{p+1,l+2} - 4\,w_{p+1,l+1} + 6\,w_{p+1,l} - 4\,w_{p+1,l-1} + w_{p+1,l-2} + w_{p,l+2} - 4\,w_{p,l+1} + 6\,w_{p,l} - 4\,w_{p,l-1} + w_{p,l-2}\big). \tag{9}$$

Substituting (5)–(9) into (2) gives, after routine manipulations,

$$\begin{aligned}&\left(\frac{1}{\delta_r^4} + \frac{1}{r\,\delta_r^3}\right)\big(w_{p+2,l+2} + w_{p+1,l+2} + w_{p,l+2}\big)\\ &+ \left(-\frac{4}{\delta_r^4} - \frac{2}{r\,\delta_r^3} - \frac{1}{r^2\,\delta_r^2} + \frac{1}{2\,r^3\,\delta_r}\right)\big(w_{p+2,l+1} + w_{p+1,l+1} + w_{p,l+1}\big)\\ &+ \left(\frac{6}{\delta_r^4} + \frac{2}{r^2\,\delta_r^2} + \frac{3\,\rho\,h}{D\,\delta_t^2}\right)\big(w_{p+2,l} + w_{p,l}\big)\\ &+ \left(\frac{6}{\delta_r^4} + \frac{2}{r^2\,\delta_r^2} - \frac{6\,\rho\,h}{D\,\delta_t^2}\right)\big(w_{p+1,l}\big)\\ &+ \left(-\frac{4}{\delta_r^4} + \frac{2}{r\,\delta_r^3} - \frac{1}{r^2\,\delta_r^2} - \frac{1}{2\,r^3\,\delta_r}\right)\big(w_{p+2,l-1} + w_{p+1,l-1} + w_{p,l-1}\big)\\ &+ \left(\frac{1}{\delta_r^4} - \frac{1}{r\,\delta_r^3}\right)\big(w_{p+2,l-2} + w_{p+1,l-2} + w_{p,l-2}\big) = \frac{3}{D}\,f_{p,l}.\end{aligned} \tag{10}$$

B. *Case* $r = 0$

Consider the following terms in (2)

$$\frac{2}{r}\frac{\partial^3 w}{\partial r^3}, \quad -\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}, \quad \frac{1}{r^3}\frac{\partial w}{\partial r}$$

at $r = 0$. Using L'Hôpital's rule gives

$$\lim_{r\to 0} \frac{2\frac{\partial^3 w}{\partial r^3}}{r} = \lim_{r\to 0} \frac{2\frac{\partial^4 w}{\partial r^4}}{1} = 2\frac{\partial^4 w}{\partial r^4}, \tag{11}$$

$$-\lim_{r\to 0} \frac{\frac{\partial^2 w}{\partial r^2}}{r^2} = -\lim_{r\to 0} \frac{\frac{\partial^3 w}{\partial r^3}}{2\,r} = -\lim_{r\to 0} \frac{\frac{\partial^4 w}{\partial r^4}}{2} = -\frac{1}{2}\frac{\partial^4 w}{\partial r^4}, \tag{12}$$

$$\lim_{r\to 0} \frac{\frac{\partial w}{\partial r}}{r^3} = \lim_{r\to 0} \frac{\frac{\partial^2 w}{\partial r^2}}{3\,r^2} = \lim_{r\to 0} \frac{\frac{\partial^3 w}{\partial r^3}}{6\,r} = \lim_{r\to 0} \frac{\frac{\partial^4 w}{\partial r^4}}{6} = \frac{1}{6}\frac{\partial^4 w}{\partial r^4}. \tag{13}$$

Hence, at $r = 0$ the PDE (2) is approximated by

$$\frac{8}{3}\frac{\partial^4 w}{\partial r^4} + \frac{\rho\,h}{D}\frac{\partial^2 w}{\partial t^2} = \frac{f}{D}. \tag{14}$$

Substituting (5) and (9) into (14) gives, after routine manipulations,

$$\begin{aligned}&\frac{8}{9\,\delta_r^4}\big(w_{p+2,l+2} + w_{p+1,l+2} + w_{p,l+2} + w_{p+2,l-1} + w_{p+1,l-1} + w_{p,l-1}\big) - \frac{32}{9\,\delta_r^4}\big(w_{p+2,l+1} + w_{p+1,l+1} + w_{p,l+1} + w_{p+2,l-2} + w_{p+1,l-2} + w_{p,l-2}\big)\\ &+ \left(\frac{16}{3\,\delta_r^4} + \frac{\rho\,h}{D\,\delta_t^2}\right)\big(w_{p+2,l} + w_{p,l}\big) + \left(\frac{16}{3\,\delta_r^4} - \frac{2\,\rho\,h}{D\,\delta_t^2}\right) w_{p+1,l} = \frac{1}{D}\,f_{p,l}.\end{aligned} \tag{15}$$

IV. NUMERICAL STABILITY ANALYSIS

A. *Case* $r > 0$

Consider (10) with zero right-hand side and substitute $g^p\,e^{j\,l\,\theta}$ for $w_{p,l}$, where $j = \sqrt{-1}$, g is termed amplification factor and θ is the spatial frequency. By von Neumann analysis [1], (10) is stable if and only if $|g| \le 1$ for all values of θ. With the above substitution, (10) can be written as a polynomial in g of the form

$$A_2(\theta)\,g^2 + A_1(\theta)\,g + A_0(\theta), \tag{16}$$

where non-constant coefficients A_0, A_1, A_2 are uniquely defined and follow immediately from (10). Then, $|g| \le 1$ if and only if

$$A_2 + A_1 + A_0 \;\ge\; 0, \tag{17}$$

$$A_2 - A_1 + A_0 \;\ge\; 0, \tag{18}$$

$$A_2 - A_0 \;\ge\; 0 \tag{19}$$

for all values of θ.

Using Euler's formula, (17) can be written as

$$\frac{2}{\delta_r^4}\cos(2\,\theta) - \left(\frac{8}{\delta_r^4} + \frac{2}{r^2\,\delta_r^2}\right)\cos(\theta) + \frac{2\,j}{r\,\delta_r^3}\sin(2\,\theta) + \left(-\frac{4}{r\,\delta_r^3} + \frac{1}{r^3\,\delta_r}\right) j\,\sin(\theta) + \left(\frac{6}{\delta_r^4} + \frac{2}{r^2\,\delta_r^2}\right) \ge 0. \tag{20}$$

A complex number is non-negative if its real part is negative and hence (20) becomes

$$\frac{2}{\delta_r^4}\cos(2\,\theta) - \left(\frac{8}{\delta_r^4} + \frac{2}{r^2\,\delta_r^2}\right)\cos(\theta) + \left(\frac{6}{\delta_r^4} + \frac{2}{r^2\,\delta_r^2}\right) \ge 0 \tag{21}$$

and hence (21) is satisfied for all values of θ.

Similarly, the condition of (18) can be written as

$$\frac{2}{3\,\delta_r^4}\cos(2\,\theta) - \left(\frac{8}{3\,\delta_r^4} + \frac{2}{3\,r^2\,\delta_r^2}\right)\cos(\theta) + \frac{2\,j}{3\,r\,\delta_r^3}\sin(2\,\theta) + \left(-\frac{4}{3\,r\,\delta_r^3} + \frac{1}{3\,r^3\,\delta_r}\right) j\,\sin(\theta) + \left(\frac{2}{\delta_r^4} + \frac{2}{3\,r^2\,\delta_r^2} + \frac{4\,\rho\,h}{D\,\delta_t^2}\right) \ge 0. \tag{22}$$

and the real part of (22) can be made non-negative for all values of θ. Also (19) is always satisfied and since (17)–(19) hold, (10) is a stable approximation when $r > 0$.

B. Case $r = 0$

In this case (17)–(19) are, respectively,

$$\frac{16}{3\,\delta_r^4}\cos(2\,\theta) - \frac{64}{3\,\delta_r^4}\cos(\theta) + \frac{16}{\delta_r^4} \geq 0 \quad (23)$$

$$\frac{16}{9\,\delta_r^4}\cos(2\,\theta) - \frac{64}{9\,\delta_r^4}\cos(\theta) + \frac{16}{3\,\delta_r^4} + \frac{4\,\rho\,h}{D\,\delta_t^2} \geq 0 \quad (24)$$

$$0 \geq 0. (25)$$

It is straightforward to show that the these conditions are always satisfied and hence (15) is a stable approximation when $r = 0$.

It follows from above results that (15) is a stable approximation of (2) for all $r \geq 0$ and all values of δ_t and δ_r. However, this does not follow that solution to this approximation for arbitrary values of parameters perfectly fits the solution to the original PDE, see later in this paper.

C. Boundary conditions

Using only discrete points of the grid (see Fig. 1), the boundary conditions (3) are

$$w_{p,l+\frac{n+1}{2}} = 0, \qquad w_{p,l+\frac{n+1}{2}+1} - w_{p,l+\frac{n+1}{2}} = 0 \qquad (26)$$

for all $p \geq 0$. Equivalently, the deflection is zero at all points marked by red and blue colours in Fig. 1 and they will not be included in the simulations given later in the paper.

V. THE STATE-SPACE MODEL OF A DEFORMABLE MIRROR

The aim of this paper is to develop a mathematical model of a deformable mirror vibrations in state-space form to facilitate control design, which requires the introduction of additional notation.

On application of the discretization scheme r can have only discrete values

$$0, \delta_r, 2\,\delta_r, \ldots, \frac{n+1}{2}\,\delta_r = \frac{a}{2}.$$

Let m be an integer and $r = m\,\delta_r$, $0 \leq m \leq \bar{m}$, where $\bar{m} = (n-1)/2$. Also the coefficients of (15) can be written as

$$P_0 = \frac{8}{9\,\delta_r^4}, \quad Q_0 = -\frac{32}{9\,\delta_r^4}, \qquad (27)$$

$$R_0 = \frac{16}{3\,\delta_r^4} + \frac{\rho\,h}{D\,\delta_t^2}, \quad \hat{R}_0 = \frac{16}{3\,\delta_r^4} - \frac{2\,\rho\,h}{D\,\delta_t^2}. \qquad (28)$$

Similarly, we write coefficients of (10) for all values $1 \leq m \leq \bar{m}$ as

$$P_m = \frac{1}{3\,\delta_r^4} + \frac{1}{3\,m\,\delta_r\,\delta_r^3}$$

$$Q_m = -\frac{4}{3\,\delta_r^4} - \frac{2}{3\,m\,\delta_r\,\delta_r^3} - \frac{1}{3\,(m\,\delta_r)^2\,\delta_r^2} + \frac{1}{6\,(m\,\delta_r)^3\,\delta_r}$$

$$R_m = \frac{2}{\delta_r^4} + \frac{2}{3\,(m\,\delta_r)^2\,\delta_r^2} + \frac{\rho\,h}{D\,\delta_t^2}$$

$$\hat{R}_m = \frac{2}{\delta_r^4} + \frac{2}{3\,(m\,\delta_r)^2\,\delta_r^2} - \frac{2\,\rho\,h}{D\,\delta_t^2}$$

$$S_m = -\frac{4}{3\,\delta_r^4} + \frac{2}{3\,m\,\delta_r\,\delta_r^3} - \frac{1}{3\,(m\,\delta_r)^2\,\delta_r^2} - \frac{1}{6\,(m\,\delta_r)^3\,\delta_r}$$

$$T_m = \frac{1}{3\,\delta_r^4} - \frac{1}{3\,m\,\delta_r\,\delta_r^3}$$

Applying lifting along the spatial variable, i.e. introducing the supervectors

$$W_p = \begin{pmatrix} w_{p,1} \\ w_{p,2} \\ \vdots \\ w_{p,n} \end{pmatrix}, \qquad F_p = \begin{pmatrix} f_{p,1} \\ f_{p,2} \\ \vdots \\ f_{p,n} \end{pmatrix} \qquad (29)$$

enables (10) to be written in matrix form as

$$A\,W_{p+2} + B\,W_{p+1} + A\,W_p = C\,F_p \qquad (30)$$

where A and B are given in (31) and (32), respectively, and

$$C = \mathrm{diag}\left(\frac{1}{D}, \ldots, \frac{1}{D}\right). \qquad (33)$$

Assuming that A is invertible (30) can be rewritten as

$$W_{p+2} = -A^{-1}\,B\,W_{p+1} - W_p + A^{-1}\,C\,F_p. \qquad (34)$$

and introducing

$$\mathbf{W}_p = \begin{bmatrix} W_{p+1} \\ W_p \end{bmatrix}, \qquad (35)$$

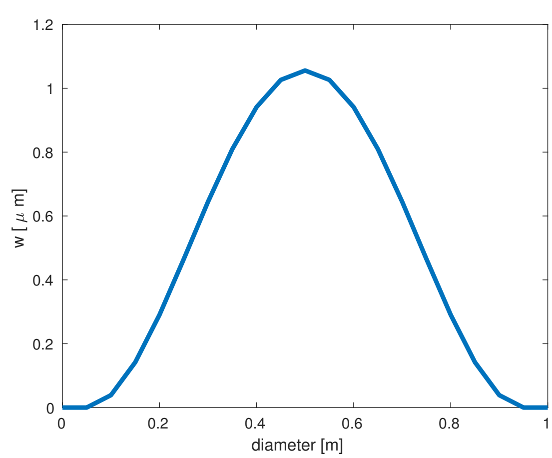

Fig. 2. Initial conditions

leading to the state equation form

$$\mathbf{W}_{p+1} = \begin{bmatrix} -A^{-1}B & -I \\ I & O \end{bmatrix} \mathbf{W}_p + \begin{bmatrix} A^{-1}C \\ O \end{bmatrix} F_p \qquad (36)$$

with $p \geq 0$.

VI. SIMULATION VERIFICATION

Consider a circular plate of diameter $a = 1$ m with $\rho = 25$ kg m^{-2}, $\nu = 0.22$, $E = 7 \cdot 10^{10}$ N m^{-2} and $h = 0.002$ m. Let $n = 9$. The responses to the initial condition given in Fig. 2 is computed, where this condition is symmetrically distributed with respect to the center of the mirror. Simulated deflections at the middle node of the plate are shown for $\delta_t = 1/1000$ s and $\delta_t = 1/2000$ s in Fig. 3. Deflections of the plate at $t = 0.96$ s for $\delta_t = 1/1000$ s and $\delta_t = 1/2000$ s are shown in Fig. 4. These responses are stable and consistent with the physics of the problem. Moreover, the responses for the both sampling time periods are very similar, confirming the correctness of the discretisation.

By the sampling theorem see, e. g. [12], the sampling period δ_t should be $f_{\mathrm{t}} = 1/\delta_t \geq 2 f_{\max}$, where $f_{\max}$ denotes the maximum frequency in the frequency response of system. To apply this theorem requires that the frequency response is bandlimited and the highest non-zero frequency must be known. In practice, various sampling periods were implemented and the choice of $f_{\max} = 500$ Hz was made. The frequency response for this sampling period is very similar to that for the sampling period used, see Fig. 3. However, the values $\delta_t = 1/50$ s or $\delta_t = 1/100$ s are too high since corresponding responses are significantly different, see Fig. 5. The response for $\delta_t = 1/2$ shown in Fig. 6, which is stable but this value is not acceptable since the oscillations in this plot are not symmetric about the origin. In conclusion, a unconditionally stable approximation of (2) for all δ_t can be constructed but the choice of parameters used must be verified by simulation. Clearly, there is further work to be done but in control systems design terms, an approximate model suitable for control design can be obtained.

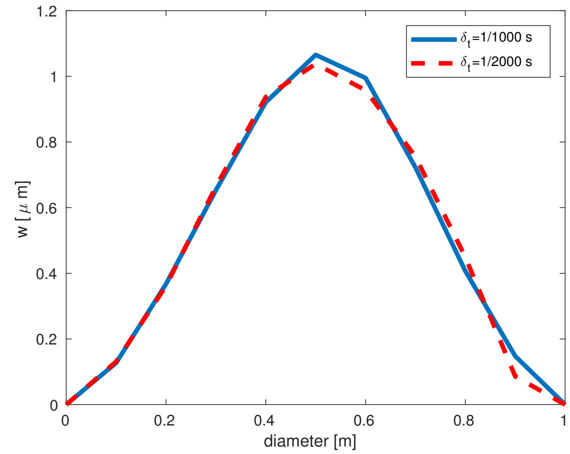

Fig. 4. Deflection of the plate at $t = 0.96$ s for $\delta_t = 1/1000$ s and $\delta_t = 1/2000$ s. Deflection is symmetrically distributed with respect to the center of the mirror.

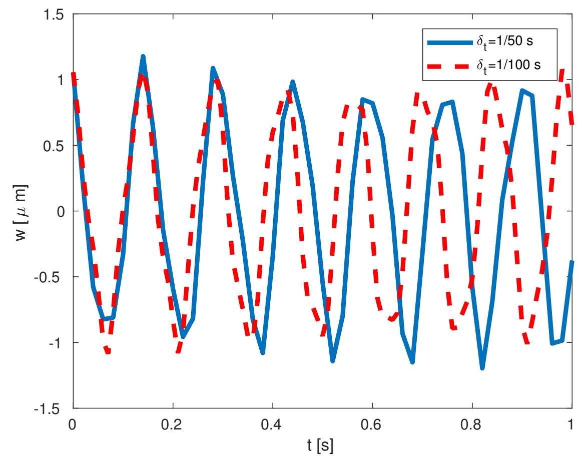

Fig. 5. Response to initial conditions at the middle node of the plate for $\delta_t = 1/50$ s and $\delta_t = 1/100$ s

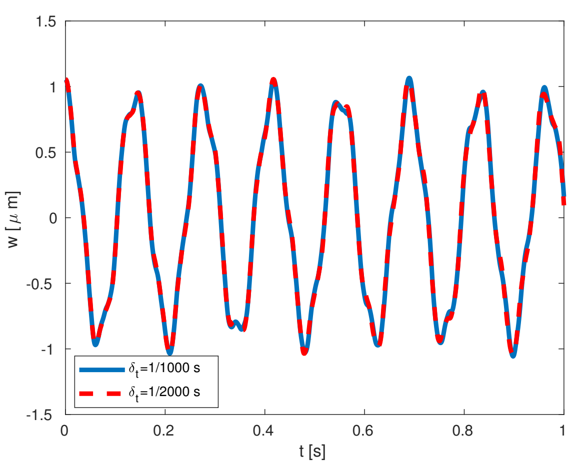

Fig. 3. Response to initial conditions at the middle node of the plate for $\delta_t = 1/1000$ s and $\delta_t = 1/2000$ s

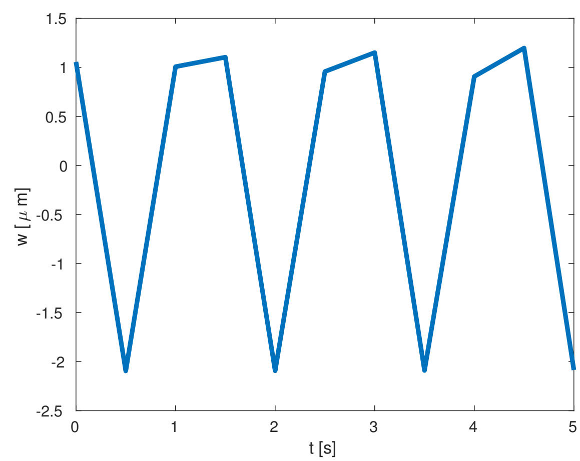

Fig. 6. Response to initial conditions at the middle node of the plate for $\delta_t = 1/2$ s

VII. CONCLUSIONS

This paper has developed a *Crank-Nicolson* based discretization scheme for systems described by a PDE in Lagrange form. In this paper, the physical problem of controlling the vibrations of a circular thin plate using an approximate finite dimensional model has been considered but the construction of this model can be applied to other examples. In contrast to previous work, discretisation using polar coordinates has been used. The unconditional stability of the approximation has been established but the sampling period implications needs further research/development.

Use of polar coordinates results in difference equations that approximately model the dynamics in one spatial indeterminate and time, unlike the rectangular (for a square system) or hexagonal a (for circular system), results in difference equations in two spatial indeterminates and time. Hence immediate benefits in terms of the dimensions of the matrices involved should this approximate model be written as a difference system in time to enable the application of standard (in time) systems theory

Future work will concentrate on the use of this model in applications, such as those to which iterative learning control can be applied. In particular , the model of (36), which is in the form of a first order discrete state-space equation, obtained by using the lifting along the spatial variable l is a good basic point for this form of control law design. One of the options is to use the repetitive processes approach that is based on introducing the increments of the state variables along the trial and an increment of the errors from trial-to-trial, see e.g., [3].

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$$A = \begin{pmatrix}
R_{\bar{m}} & S_{\bar{m}} & T_{\bar{m}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
Q_{\bar{m}-1} & R_{\bar{m}-1} & S_{\bar{m}-1} & T_{\bar{m}-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
P_{\bar{m}-2} & Q_{\bar{m}-2} & R_{\bar{m}-2} & S_{\bar{m}-2} & T_{\bar{m}-2} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & P_1 & Q_1 & R_1 & S_1 & T_1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & P_0 & Q_0 & R_0 & Q_0 & P_0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & P_1 & Q_1 & R_1 & S_1 & T_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & P_{\bar{m}-2} & Q_{\bar{m}-2} & R_{\bar{m}-2} & S_{\bar{m}-2} & T_{\bar{m}-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & P_{\bar{m}-1} & Q_{\bar{m}-1} & R_{\bar{m}-1} & S_{\bar{m}-1} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & P_{\bar{m}} & Q_{\bar{m}} & R_{\bar{m}}
\end{pmatrix} \quad (31)$$

$$B = \begin{pmatrix}
\hat{R}_{\bar{m}} & S_{\bar{m}} & T_{\bar{m}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
Q_{\bar{m}-1} & \hat{R}_{\bar{m}-1} & S_{\bar{m}-1} & T_{\bar{m}-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
P_{\bar{m}-2} & Q_{\bar{m}-2} & \hat{R}_{\bar{m}-2} & S_{\bar{m}-2} & T_{\bar{m}-2} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & P_1 & Q_1 & \hat{R}_1 & S_1 & T_1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & P_0 & Q_0 & \hat{R}_0 & Q_0 & P_0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & P_1 & Q_1 & \hat{R}_1 & S_1 & T_1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & P_{\bar{m}-2} & Q_{\bar{m}-2} & \hat{R}_{\bar{m}-2} & S_{\bar{m}-2} & T_{\bar{m}-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & P_{\bar{m}-1} & Q_{\bar{m}-1} & \hat{R}_{\bar{m}-1} & S_{\bar{m}-1} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & P_{\bar{m}} & Q_{\bar{m}} & \hat{R}_{\bar{m}}
\end{pmatrix} \quad (32)$$