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Matrix Codes With the Rank-Metric and Their Covering Radii

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***Abstract*— The covering radius of a rank-metric code is defined as the maximum distance between the code and a matrix from the ambient space. This fundamental parameter measures the performance of a code both in error correction and source coding applications. In this paper we discuss some structural properties of matrix codes endowed with the rank metric, and relate them to the covering radius. In particular, we derive upper bounds on the covering radius of a rank-metric code by applying different combinatorial methods.**

I. INTRODUCTION

In recent years, rank-metric codes have featured prominently in the Coding Theory literature, especially after their applications to error control in network communications were understood. A rank-metric code is a subset of the matrix space $\mathbb{F}_q^{k\times m}$ endowed with the rank distance function.

There are only few known classes of rank-metric codes [10], [11], [20], which are optimal and can be efficiently decoded [11], [15], [23].

In this paper, we study the *covering radius* of a (matrix) rank-metric code. This fundamental parameter measures the maximum weight of any correctable error in the ambient space. It also characterizes the *maximality* property of a code, that is, whether or not the code is contained in a supercode of the same minimum distance. The covering radius of a code can also be viewed as the least integer r such that every element of the ambient space is within distance r of some codeword.

There are numerous papers and books on this topic for classical codes with respect to the Hamming metric (see [1], [4], [5], [6], [14] and the references therein), but relatively little attention has been paid to it for (matrix) rank-metric codes [12], [13].

In this paper we develop combinatorial tools to derive upper bounds on the covering radius of (not necessarily linear) rank-metric codes. Some of the derived bounds, such as the dual distance and external distance bounds, can be seen as analogues of bounds for the Hamming metric. Others, such as the initial set bound, are unique to matrix codes.

The remainder of the paper is organized as follows: Section II is devoted to preliminary concepts and results on rank-metric codes. In Section III we consider the property of *maximality*. We say that a code is maximal if it is not contained in a proper supercode of the same minimum distance. We introduce a new parameter, called the *maximality degree* of a code, and show that it is determined by the minimum distance and the covering radius of the code. In Section IV we investigate translates of a code, showing that the weight enumerator of a coset of a linear code is completely determined by the weights of first $n-d^\perp$ cosets. We establish this result using Möbius inversion on the lattice of subspaces of $\mathbb{F}_q^k$, and apply it to derive a *dual distance bound* on the covering radius of a linear rank-metric code. In section V we give the rank-metric analogue of the *external distance bound*, which holds also for non-linear codes. In Section VI we then introduce the concept of the *initial set* of a matrix code, and use this to derive a third bound on the covering radius of a code.

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II. RANK-METRIC CODES

In the sequel q is a fixed prime power, and $\mathbb{F}_q$ is the finite field with q elements. We also fix positive integers $k\le m$ and denote by $\mathbb{F}_q^{k\times m}$ the space of $k\times m$ matrices over $\mathbb{F}_q$. For any integer $n\ge 1$ we set $[n]:=\{i\in\mathbb{N}:1\le i\le n\}$.

Definition 1. The **rank distance** between matrices $M,N\in\mathbb{F}_q^{k\times m}$ is $d(M,N):=\mathrm{rk}(M-N)$. A **rank-metric code** is a non-empty subset $\mathcal{C}\subseteq\mathbb{F}_q^{k\times m}$. If $|\mathcal{C}|\ge 2$, then the **minimum distance** of $\mathcal{C}$ is the integer

$$d(\mathcal{C}):=\min\{d(M,N):M,N\in\mathcal{C},\ M\neq N\}.$$

The **weight** and **distance distribution** of $\mathcal{C}\subseteq\mathbb{F}_q^{k\times m}$ are the integer vectors $W(\mathcal{C})=(W_i(\mathcal{C}):0\le i\le k)$ and $B(\mathcal{C})=(B_i(\mathcal{C}):0\le i\le k)$, where, for all i,

$$\begin{aligned} W_i(\mathcal{C}) &:= |\{M\in\mathcal{C}:\mathrm{rk}(M)=i\}|,\\ B_i(\mathcal{C}) &:= |\mathcal{C}|^{-1}\cdot|\{(M,N)\in\mathcal{C}^2:d(M,N)=i\}|. \end{aligned}$$

The **dual code** of a linear code $\mathcal{C}$ is

$$\mathcal{C}^\perp:=\{N\in\mathbb{F}_q^{k\times m}:\mathrm{Tr}(MN^t)=0 \text{ for all } M\in\mathcal{C}\}.$$

If $\mathcal{C}\subseteq\mathbb{F}_q^{k\times m}$ is linear, then we have

$$d(\mathcal{C})=\min\{\mathrm{rk}(M):M\in\mathcal{C},\ M\neq 0\}$$

and $W_i(\mathcal{C})=B_i(\mathcal{C})$ for all $i\in\{0,...,k\}$. Moreover, since the map $(M,N)\mapsto\mathrm{Tr}(MN^t)$ defines an inner product on the space $\mathbb{F}_q^{k\times m}$, we have $\dim(\mathcal{C}^\perp)=km-\dim(\mathcal{C})$ and $\mathcal{C}^{\perp\perp}=\mathcal{C}$.

In this paper we study the following fundamental parameter of a rank-metric code.

Definition 2. The **covering radius** of a code $\mathcal{C}\subseteq\mathbb{F}_q^{k\times m}$ is the integer

$$\rho(\mathcal{C}):=\min\{i\in\mathbb{N}: \text{ for all } X\in\mathbb{F}_q^{k\times m} \text{ there exists } M\in\mathcal{C} \text{ with } d(X,M)\le i\}.$$

The covering radius of a code $\mathcal{C}$ is therefore the maximum distance of $\mathcal{C}$ to any matrix in the ambient space, or the minimum value r such that the union of the spheres of radius r about each codeword cover the ambient space.

The following result summarizes some simple properties of this fundamental invariant.

Lemma 3. Let $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ be a code. The following hold.
1) $\rho(\mathcal{C}) = 0$ if and only if $\mathcal{C} = \mathbb{F}_q^{k\times m}$.
2) If $\mathcal{D} \subseteq \mathbb{F}_q^{k\times m}$ is a code with $\mathcal{C} \subseteq \mathcal{D}$, then $\rho(\mathcal{C}) \geq \rho(\mathcal{D})$.
3) If $\mathcal{D} \subseteq \mathbb{F}_q^{k\times m}$ is a code with $\mathcal{C} \subsetneq \mathcal{D}$, then $\rho(\mathcal{C}) \geq d(\mathcal{D})$.
4) $d(\mathcal{C}) - 1 < 2\rho(\mathcal{C})$, if $|\mathcal{C}| \geq 2$ and $\mathcal{C} \subsetneq \mathbb{F}_q^{k\times m}$.

All the above properties are simple consequences of the definitions.

III. MAXIMALITY

In this section we investigate the connection between the covering radius of a rank-metric code and the property of maximality. A code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ is **maximal** if $|\mathcal{C}| = 1$ or $|\mathcal{C}| \geq 2$ and there is no code $\mathcal{D} \subseteq \mathbb{F}_q^{k\times m}$ with $\mathcal{D} \supsetneq \mathcal{C}$ and $d(\mathcal{D}) = d(\mathcal{C})$. In particular, $\mathbb{F}_q^{k\times m}$ is maximal.

Proposition 4 (see e.g. [5])**.** A code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ with $|\mathcal{C}| \geq 2$ is maximal if and only if $\rho(\mathcal{C}) \leq d(\mathcal{C}) - 1$.

We propose a natural parameter that measures the maximality of a code, and show how it relates to its covering radius.

Definition 5. The **maximality degree** of a code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ with $|\mathcal{C}| \geq 2$ is the integer

$$\mu(\mathcal{C}) := \min\{d(\mathcal{C}) - d(\mathcal{D}) : \mathcal{D} \subseteq \mathbb{F}_q^{k\times m} \text{ is a code with } \mathcal{D} \supsetneq \mathcal{C}\}.$$

We also put $\mu(\mathbb{F}_q^{k\times m}) := 1$.

The maximality degree of a code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ with $|\mathcal{C}| \geq 2$ satisfies $0 \leq \mu(\mathcal{C}) \leq d(\mathcal{C}) - 1$. Moreover, it is easy to see that $\mu(\mathcal{C}) > 0$ if and only if $\mathcal{C}$ is maximal. Notice that $\mu(\mathcal{C})$ can be interpreted as the minimum price (in terms of minimum distance) that one has to pay in order to enlarge $\mathcal{C}$ to a bigger code. We can state a precise relation between the covering radius and the maximality degree of a code as follows.

Proposition 6 ([2], Proposition 7)**.** For any code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ with $|\mathcal{C}| \geq 2$ we have $\mu(\mathcal{C}) = d(\mathcal{C}) - \min\{\rho(\mathcal{C}),\ d(\mathcal{C})\}$. In particular, if $\mathcal{C}$ is maximal then $\mu(\mathcal{C}) = d(\mathcal{C}) - \rho(\mathcal{C})$.

IV. TRANSLATES OF A RANK-METRIC CODE AND DUAL DISTANCE BOUND

In this section we study the weight distribution of the translates of a code. As an application, we obtain an upper bound on its covering radius.

Recall that the **translate** of a code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ by a matrix $X \in \mathbb{F}_q^{k\times m}$ is the code

$$\mathcal{C} + X := \{M + X : M \in \mathcal{C}\} \subseteq \mathbb{F}_q^{k\times m}.$$

Clearly, full knowledge of the weight distribution of the translates of $\mathcal{C}$ tells us the covering radius, which is the maximum of the minimum weight of each translate of $\mathcal{C}$. Even partial information may yield a bound on the covering radius. More precisely, if $X \in \mathbb{F}_q^{k\times m}$ and $W_i(\mathcal{C} + X) \neq 0$, then $d(X, \mathcal{C}) := \min\{d(X, M) : M \in \mathcal{C}\} \leq i$. So if there exists r such that for each $X \in \mathbb{F}_q^{k\times m}$, $W_i(\mathcal{C} + X) \neq 0$ for some $i \leq r$ then, in particular, $\rho(\mathcal{C}) \leq r$. If such a value r can be determined, then we get an upper bound on the covering radius of $\mathcal{C}$.

The goal of this section is twofold. We first show that the weight distribution $W_0(\mathcal{C} + X), ..., W_k(\mathcal{C} + X)$ of the translate $\mathcal{C} + X$ of a linear code $\mathcal{C} \subsetneq \mathbb{F}_q^{k\times m}$ is determined by the values of $W_0(\mathcal{C} + X), ..., W_{k-d^\perp}(\mathcal{C} + X)$, where $d^\perp = d(\mathcal{C}^\perp)$. Moreover, we provide explicit formulas for $W_{k-d^\perp+1}(\mathcal{C} + X), ..., W_k(\mathcal{C} + X)$ as linear functions of $W_0(\mathcal{C} + X), ..., W_{k-d^\perp}(\mathcal{C} + X)$.

In a second part, we obtain an upper bound on the covering radius of a linear code in terms of the distance of its dual code.

Theorem 7 ([2], Theorem 20)**.** Let $\mathcal{C} \subsetneq \mathbb{F}_q^{k\times m}$ be linear, and let $X \in \mathbb{F}_q^{k\times m}$ be any matrix. Write $d^\perp := d(\mathcal{C}^\perp)$. Then for all integers $k - d^\perp + 1 \leq i \leq k$ we have

$$W_i(\mathcal{C} + X) = \sum_{u=0}^{k-d^\perp} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} k-u \\ i-u \end{bmatrix}_q \cdot \sum_{j=0}^{u} W_j(\mathcal{C} + X) \begin{bmatrix} k-j \\ u-j \end{bmatrix}_q + \sum_{u=k-d^\perp+1}^{i} \begin{bmatrix} k \\ u \end{bmatrix}_q \frac{|\mathcal{C}|}{q^{m(k-u)}}.$$

In particular, the distance distribution of the translate code $\mathcal{C}+X$ is completely determined by k, m, $|\mathcal{C}|$ and the weights $W_0(\mathcal{C} + X), ..., W_{k-d^\perp}(\mathcal{C} + X)$.

As a simple consequence of Theorem 7 we obtain an upper bound on the covering radius of a linear code $\mathcal{C} \subsetneq \mathbb{F}_q^{k\times m}$ in terms of its dual distance. Let $X \in \mathbb{F}_q^{k\times m} \notin \mathcal{C}$ be an arbitrary matrix. Then we have $W_0(\mathcal{C}+X) = 0$. Now Theorem 7 with $i := k - d^\perp + 1$ gives

$$W_{k+d^\perp+1}(\mathcal{C} + X) = \sum_{u=1}^{k-d^\perp} (-1)^{i-u} q^{\binom{i-u}{2}} \begin{bmatrix} k-u \\ i-u \end{bmatrix}_q \cdot \sum_{j=1}^{u} W_j(\mathcal{C}+X) \begin{bmatrix} k-j \\ u-j \end{bmatrix}_q + \begin{bmatrix} k \\ k-d^\perp+1 \end{bmatrix}_q |\mathcal{C}|/q^{m(d^\perp-1)}.$$

In particular, $W_1(\mathcal{C} + X), ..., W_{k-d^\perp+1}(\mathcal{C} + X)$ cannot be all zero.

Corollary 8 (dual distance bound, Corollary 21 of [2])**.** For a linear code $\mathcal{C} \subsetneq \mathbb{F}_q^{k\times m}$ we have $\rho(\mathcal{C}) \leq k - d(\mathcal{C}^\perp) + 1$.

V. EXTERNAL DISTANCE BOUND

Following work of Delsarte for the Hamming metric [8], in this section we apply Fourier transform methods to obtain further results on the weight distributions of the translates of a (not necessarily linear) code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$. In particular, we obtain an upper bound for the covering radius of a general rank-metric code in terms of its external distance.

Throughout the reminder of this section $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ denotes a (possibly non-linear) code, and χ is a fixed non-trivial character of $(\mathbb{F}_q, +)$.

Definition 9. Let $Y \in \mathbb{F}_q^{k\times m}$. Define the **character map** on $(\mathbb{F}_q^{k\times m}, +)$ associated to Y by

$$\phi_Y : \mathbb{F}_q^{k\times m} \to \mathbb{C}^\times : X \mapsto \chi(\mathrm{Tr}\,(YX^T)).$$

Clearly $\phi_X(Y) = \phi_Y(X)$ for all $X, Y \in \mathbb{F}_q^{k\times m}$. We let Φ denote the $km \times km$ symmetric matrix with values in $\mathbb{C}^\times$ defined as having entry $\phi_Y(X)$ in the column indexed by X and in the row indexed by Y. Define the $\mathbb{Q}$-module of length km: $\mathfrak{C} := \left\{(\mathcal{A}_X : X \in \mathbb{F}_q^{k\times m}) : \mathcal{A}_X \in \mathbb{Q}\right\}$. For each Y, extend ϕ_Y to a character of $\mathfrak{C}$ as follows:

$$\phi_Y : \mathfrak{C} \to \mathbb{C}^\times : \mathcal{A} = (\mathcal{A}_X : X \in \mathbb{F}_q^{k\times m}) \mapsto \sum_X \mathcal{A}_X \phi_Y(X).$$

Then $\Phi\mathcal{A} = (\phi_Y(\mathcal{A}) : Y \in \mathbb{F}_q^{k\times m}) \in \mathfrak{C}$. It can be shown that the rows of Φ are pairwise orthogonal. Therefore $\Phi^2\mathcal{A} = \Phi^T\Phi\mathcal{A} = q^{km}\mathcal{A}$, and so $\mathcal{A}$ is determined completely by its transform

$$\mathcal{A}^* := \Phi\mathcal{A} = (\phi_Y(\mathcal{A}) : Y \in \mathbb{F}_q^{k\times m}).$$

Any subset $\mathcal{U} \subseteq \mathbb{F}_q^{k\times m}$ can be identified with the 0-1 vector $\overline{\mathcal{U}} = (\mathcal{U}_Z : Z \in \mathbb{F}_q^{k\times m}) \in \mathfrak{C}$, where

$$\mathcal{U}_Z = \begin{cases} 1 & \text{if } Z \in \mathcal{U}, \\ 0 & \text{otherwise.} \end{cases}$$

For any $X \in \mathbb{F}_q^{k\times m}$, the translate code $\mathcal{C} + X \subseteq \mathbb{F}_q^{k\times m}$ is then identified with $\overline{\mathcal{C}+X} = (\mathcal{C}_{Z-X} : Z \in \mathbb{F}_q^{k\times m})$. It is straightforward to show that $\phi_Y(\overline{\mathcal{C}+X}) = \phi_Y(\overline{\mathcal{C}})\phi_Y(X)$. This immediately yields the inversion formula

$$\mathcal{C}_X = \frac{1}{q^{km}}\sum_Y \phi_Y(\overline{\mathcal{C}+X}) = \frac{1}{q^{km}}\sum_Y \phi_Y(\overline{\mathcal{C}})\phi_Y(X).$$

For each $i \in [k]$ we let Ω^i be the set of matrices in $\mathbb{F}_q^{k\times m}$ of rank i.

Claim (see [10])**.** Let $Y \in \mathbb{F}_q^{k\times m}$. Then $\phi_Y(\overline{\Omega^i})$ depends only on the rank of Y. If Y has rank j, then this is given by

$$P_i(j) := \sum_{\ell=0}^{k} (-1)^{i-\ell} q^{\ell m + \binom{i-\ell}{2}} \begin{bmatrix} k-\ell \\ k-i \end{bmatrix}_q \begin{bmatrix} k-j \\ \ell \end{bmatrix}_q.$$

In terms of the transform of Ω^i, the claim gives

$$\Phi\overline{\Omega^i} = (P_i(\mathrm{rk}(Y)) : Y \in \mathbb{F}_q^{k\times m}).$$

It is known [9], [10] that the $P_i(j)$ are orthogonal polynomials of degree i in the variable q^{-j}. Therefore, any rational polynomial γ of degree at most k in q^{-j} can be expressed as a $\mathbb{Q}$-linear combination of the q-Krawtchouck polynomials: $\gamma(x) = \sum_{j=0}^{k} \gamma_j P_j(x)$. Again, the orthogonality relations mean that the coefficients can be of γ can be retrieved as

$$\gamma_j = \frac{1}{q^{km}} \sum_{i=0}^{k} \gamma(i) P_i(j).$$

We let $P = (P_i(j))$ denote the $(k+1)\times(k+1)$ matrix with (j,i)-th component equal to $P_i(j)$. Then the **transform** of $B(\mathcal{C}) = (B_i(\mathcal{C}) : 0 \le i \le k)$ is defined as

$$B^*(\mathcal{C}) := |\mathcal{C}|^{-1} B(\mathcal{C}) P.$$

The coefficents of $B^*(\mathcal{C})$ are non-negative (see e.g. [10, Theorem 3.2]).

Definition 10. The **external distance** of a code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ is the integer

$$\sigma^*(\mathcal{C}) := |\{1 \le i \le k : B_i^*(\mathcal{C}) > 0\}|.$$

We can now upper-bound the covering radius of a general rank-metric code in terms of its external distance as follows.

Theorem 11 (external distance bound, Theorem 27 of [2])**.** For any code $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ we have $\rho(C) \le \sigma^*(\mathcal{C})$.

VI. INITIAL SET BOUND

In this section we propose a definition of initial set of a linear rank-metric code inspired by [17]. Moreover we exploit the combinatorial structure of such set to derive an upper bound for the covering radius of the underlying code.

Note that our technique to derive the bound relies on the specific "matrix structure" of rank-metric codes.

Notation 12. For positive integers a, b and a set $S \subseteq [a]\times[b]$, we denote by $\mathbb{I}(S) \in \mathbb{F}_2^{a\times b}$ be the binary matrix defined by $\mathbb{I}(S)_{ij} := 1$ if $(i,j) \in S$, and $\mathbb{I}(S) := 0$ if $(i,j) \notin S$. Moreover, we denote by $\lambda(S)$ the minimum number of lines (rows or columns) required to cover all the ones in $\mathbb{I}(S)$.

The initial set of a linear code is defined as follows.

Definition 13. Denote by $\preceq$ the lexicographic order on the set $[k]\times[m]$. The **initial entry** of a non-zero matrix $M \in \mathbb{F}_q^{k\times m}$ is $\mathrm{in}(M) := \min_{\preceq}\{(i,j) : M_{ij} \neq 0\}$. The **initial set** of a non-zero linear code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ is

$$\mathrm{in}(\mathcal{C}) := \{\mathrm{in}(M) : M \in \mathcal{C},\ M \neq 0\}.$$

We start with a preliminary lemma that summarizes two important properties of the initial set of a code.

Lemma 14. Let $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ be a non-zero linear code. The following hold.

1) $\dim(\mathcal{C}) = |\mathrm{in}(\mathcal{C})|$,
2) $\mathrm{in}(\mathcal{C}) \subseteq [k - d(\mathcal{C}) + 1] \times [m]$.

We can now state the main result of this section, which provides an upper bound on the covering radius of a linear rank-metric code $\mathcal{C}$ in terms of the combinatorial structure of its initial set.

Theorem 15 (initial set bound, Theorem 34 of [2])**.** Let $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ be a non-zero linear code. We have

$$\rho(\mathcal{C}) \le d(\mathcal{C}) - 1 + \lambda(S),$$

where $S := [k - d(\mathcal{C}) + 1] \times [m] \setminus \mathrm{in}(\mathcal{C})$.

Remark 16. The initial set of a linear code $\mathcal{C} \subseteq \mathbb{F}_q^{k\times m}$ can be efficiently computed from any basis of $\mathcal{C}$ as follows. Let

$w : \mathbb{F}_q^{k\times m} \to \mathbb{F}_q^{mk}$ denote the "vectorization" map that sends a matrix M to the mk-vector obtained by concatenating the rows of M. Now given a basis $\{M_1, ..., M_t\}$ of $\mathcal{C}$, construct the vectors $v_1 := w(M_1), ..., v_t := w(M_t)$. Then perform Gaussian elimination on $\{v_1, ..., v_t\}$ and obtain vectors $\overline{v}_1, ..., \overline{v}_t$. Clearly, $\{w^{-1}(\overline{v}_1), ..., w^{-1}(\overline{v}_t)\}$ is still a basis of $\mathcal{C}$, and one can check that

$$\mathrm{in}(\mathcal{C}) = \{\mathrm{in}(w^{-1}(\overline{v}_1)), ..., \mathrm{in}(w^{-1}(\overline{v}_t))\}.$$

The following example shows that Theorem 15 gives in some cases a better bound than Corollary 11 for the covering radius of a linear code.

Example 17. Let $q = 2$ and $k = m = 3$. Denote by $\mathcal{C}$ the code generated over $\mathbb{F}_2$ by the matrices

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \quad \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 1 & 0 & 0\end{bmatrix}.$$

We have $\dim(\mathcal{C}) = 4$ and $d(\mathcal{C}) = 2$. Moreover, since

$$\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} \in \mathcal{C}^\perp,$$

we have $\sigma(\mathcal{C}^\perp) = 3$, and so Corollary 11 gives $\rho(\mathcal{C}) \leq 3$. On the other hand, the initial set of $\mathcal{C}$ can be computed as $\mathrm{in}(\mathcal{C}) = \{(1,1),(1,2),(2,1),(2,2)\}$. Thus following the notation of Theorem 15 we have $S = \{(1,3),(2,3)\}$ and $\lambda(S) = 1$. It follows that $\rho(\mathcal{C}) \leq d(\mathcal{C}) - 1 + \lambda(S) = 2$. Therefore Theorem 15 gives a better bound on $\rho(\mathcal{C})$ than Corollary 11. In fact, one can check that $\rho(\mathcal{C}) = 2$.

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