23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

Roots of Polynomials with Positive Coefficients

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***Abstract*— A necessary condition for stability of a finite-dimensional linear time-invariant system is that all the coefficients of the characteristic equation are strictly positive. However, it is well-known that this condition is not sufficient, except for n less than 3. In this paper, we show that any polynomial that has positive coefficients cannot have roots on the nonnegative real axis. Conversely, if a polynomial has no roots on the positive real axis, a polynomial with positive coefficients can be found so that the product of the two polynomials also has positive coefficients. A simple upper bound for the degree of this multiplier polynomial is given. One application of the main result is that under a strict condition, it is possible to find a non-minimal realization of a given transfer function using only positive multipliers (except for the "minus" in the standard feedback comparator).**

Keywords: positive polynomial, root locations, positive-real, positive system
AMS Classification: 12D10, 26C10, 93D99

I. INTRODUCTION

All coefficients of the polynomial

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

(Example 3.30 in [8]) are positive. Positivity of the coefficients is a necessary condition to have all its roots in the open left half plane. This condition is not sufficient, as is the case in the above polynomial. By performing the Routh-Hurwitz test, it can be seen that this polynomial will have two roots in the right-half plane.
This prompts the question as to where precisely the roots of a polynomial can or cannot lie. This short note resolves this question by identifying such a root property. It also goes beyond, and shows that any polynomial satisfying this root property is a *factor* of a polynomial with all positive coefficients.

Some results related to this problem appeared in [2]. The authors asked the question if a conjugate pair of zeros can be factored out from a polynomial with nonnegative coefficients so that the resulting polynomial still has nonnegative coefficients. Sendov [14] applies their work to prove a Gauss-Lucas type theorem for non convex (symmetric w.r.t. $\mathbb{R}$) sectors in $\mathbb{C}$. Much earlier, Obrechkoff proved an upper bound on the number of roots in the sector $\{s \in \mathbb{C} \setminus \{0\} | \arg(s)| < \theta\}$ for $\theta \in (0, \pi/2)$. The bound implies that no roots can lie on $\mathbb{R}_+$. Handelman [9] showed that for a monic (Laurent) polynomial, $p \in \mathbb{R}[s]$, with non-negative roots, there exists a positive integer m such that $(1+s)^m p(s)$ has only positive coefficients. A short proof of this fact is given by Akiyama [1]. In this paper we give an easily computable bound for m and show its asymptotic behavior as the roots of p get closer to the real axis. Dubickas [7] solves an algebraic number theoretic problem, for which he proved the auxiliary lemma:
Lemma [7]: Let $s_0 \in \mathbb{C} \setminus \mathbb{R}$. Then there exists a monic polynomial $q \in \mathbb{Q}[s]$ such that the polynomial $(s - s_0)(s - \overline{s}_0)q(s)$ has positive coefficients.
This settled a conjecture by Kuba [11]. This problem is further studied by Borel [3], Zaïmi [15], and Brunotte. In [4], Brunotte finds for p of second degree, the lowest degree, δ, of q such that pq has positive coefficients and the lowest degree, δ_0, of a polynomial q_0 such that pq_0 has nonnegative coefficients, thus extending a result of Meissner [12], who constrained q to have positive coefficients. Brunotte shows that these polynomials can be calculated in finitely many steps. The exact formula for δ when $deg(p) > 2$ is still unknown. In [5] it is shown that if $p \in \mathbb{Z}[s]$, a polynomial $q \in \mathbb{Z}[s]$ exists. Our contribution is to determine a readily computable polynomial although not of minimal degree that will solve the problem, make connections with positive realness, and look at possible applications to positive system theory.

A. *Notation for some important point sets*

If $x \in \mathbb{C}$, then its real and imaginary parts are denoted by respectively $\Re(x)$ and $\Im(x)$.

$$\begin{aligned}
\mathbb{R}_- &= \{x \in \mathbb{R} | x < 0\} \\
\mathbb{R}_+ &= \{x \in \mathbb{R} | x > 0\} \\
\mathbb{C}_- &= \{x \in \mathbb{C} | \Re(x) < 0\} \\
\mathbb{C}+ &= \{x \in \mathbb{C} | \Re(x) > 0\} \\
\mathbb{C}_0 &= \{x \in \mathbb{C} | \Re(x) = 0\} \\
\mathbb{C}_{s-} &= \{x \in \mathbb{C} | \Re(x) < 0\} \setminus \mathbb{R}_- \\
\mathbb{C}_{s+} &= \{x \in \mathbb{C} | \Re(x) > 0\} \setminus \mathbb{R}_+ \\
\mathbb{C}_{s0} &= \{x \in \mathbb{C} | \Re(x) = 0\} \setminus \{0\}
\end{aligned}$$

The closure of these sets (whenever appropriate) will be denoted by an overbar.
With these number sets, the complex plane $\mathbb{C}$ partitions as

$$\mathbb{C} = \{0\} \cup \mathbb{R}_- \cup \mathbb{R}_+ \cup \mathbb{C}_{s-} \cup \mathbb{C}_{s+} \cup \mathbb{C}_{s0}.$$

*This work was supported by NSF grant: CPS 1544857
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B. Notation for some important sets of polynomials

Let $K[s]$ denote the set of polynomials in the indeterminate s, with coefficients in the set K. In particular, denote

- $\mathbb{R}_+[s]$: Set of polynomials in indeterminate s with strict positive coefficients.
- $\widehat{\mathbb{R}}_+[s]$: Set of monic polynomials in indeterminate s with strictly positive coefficients.
- $\overline{\mathbb{R}}_+[s]$: Set of polynomials in indeterminate s with non-negative coefficients.

The subset of polynomials in $K[s]$ that are *monic*, i.e., the leading degree coefficient is 1, will be denoted by $\widehat{K}[x]$.

Define also the set of polynomials in $\mathbb{R}[s]$ that are strictly *positive valued* for $s \in \mathbb{R}_+$.

$$\mathcal{P}[s] = \{\, p \in \mathbb{R}[s] \,|\, p(\mathbb{R}_+) \subset \mathbb{R}_+ \,\}.$$

Likewise, denote the polynomials that are nonnegative valued on $\overline{\mathbb{R}}_+$ by

$$\overline{\mathcal{P}}[s] = \{\, p \in \overline{\mathbb{R}}[s] \,|\, p(\overline{\mathbb{R}}_+) \subset \overline{\mathbb{R}}_+ \,\}.$$

Finally, let the polynomial sets $\mathcal{R}[x]$ and $\overline{\mathcal{R}}[x]$ respectively be defined by

$$\mathcal{R}[s] = \{\, r \in \mathbb{R}[s] \,|\, \exists q \in \mathbb{R}_+[s], \text{ s.t. } rq \in \mathbb{R}_+[s] \,\},$$

and

$$\overline{\mathcal{R}}[s] = \{\, r \in \mathbb{R}[s] \,|\, \exists q \in \mathbb{R}_+[s], \text{ s.t. } rq \in \overline{\widehat{\mathbb{R}}}_+[s] \,\}.$$

The multiplier polynomial q in the above definitions will be called an *auxiliary* polynomial for r.

II. Algebraic Structure

Let $+$ and $\times$ indicate the usual addition and multiplication of polynomials. It is readily seen that $(\mathbb{R}_+[x], +)$ is a semigroup and $(\overline{\mathbb{R}}_+[x], +)$ a monoid with the zero polynomial as neutral element. Likewise, $(\mathbb{R}_+[x], \times)$, $(\overline{\mathbb{R}}_+[x], \times)$, $(\widehat{\mathbb{R}}_+[x], \times)$, and $(\overline{\widehat{\mathbb{R}}}_+[x], \times)$ are semigroups with neutral element 1. It follows that the property of having "positive coefficients" is preserved under addition and multiplication. This will be useful in the proof of Theorem 2 below. We also note that if $r \in \mathcal{R}[s]$, the auxiliary polynomial q of r is not unique. Indeed if $p \in \mathbb{R}_+[s]$, then $pq \in \mathbb{R}_+[s]$ and $r(pq) = p(rq) \in \mathbb{R}_+[s]$ by the semigroup property. The same holds for $\overline{\mathcal{R}}[s]$.
Even if q is restricted to be monic and of lowest degree, it may not be unique. This can be shown by (counter)-example.

Since $\mathbb{C}$ is a closed field, the fundamental theorem of algebra implies that if $a \in \mathbb{C}[s]$ has degree n, it has n roots (counting multiplicity) in $\mathbb{C}$. The set of roots of the polynomial will be denoted by $\Sigma(a)$. Since this is a set it does not give information about the multiplicity of each root. If $a \in \mathbb{R}[s]$, its roots are either real or occur in complex conjugate pairs.

Let now the monic polynomial, $a \in \mathbb{R}[s]$, have k_- roots in $\mathbb{R}_-$, k_+ roots in $\mathbb{R}_+$, root 0 with multiplicity k_0, ℓ_- complex conjugate root pairs in $\mathbb{C}_-$, ℓ_+ conjugate root pairs in $\mathbb{C}_+$, and ℓ_0 conjugate root pairs in $\mathbb{C}_{s0}$, the strict imaginary axis. Then the polynomial $a(s)$ may be factored as $a_+(s)a_-(s)a_0(s)\widehat{a}(s)$, where each of the factors are monic and

$$\begin{aligned}
\Sigma(a_+) &\subset \mathbb{R}_- \cup \mathbb{C}_{s-} = \mathbb{C}_- \\
\Sigma(a_-) &\subset \mathbb{C}_{s+} \\
\Sigma(a_0) &\subset \{0\} \cup \mathbb{C}_{s0} = \mathbb{C}_0 \\
\Sigma(\widehat{a}) &\subset \mathbb{R}_+.
\end{aligned}$$

Furthermore,

$$a_+(s) = \prod_{i=1}^{k_-}(s + r_i)\prod_{i=1}^{\ell_-}(s^2 + 2\sigma_i s + (\sigma_i^2 + \omega_i^2)), \quad (1)$$

$$r_i > 0,\ \sigma_i > 0,\ \omega_i > 0. \quad (2)$$

$$a_-(s) = \prod_{i=1}^{\ell_+}(s^2 - 2\xi_i s + (\xi_i^2 + \eta_i^2)), \quad (3)$$

$$\xi_i > 0,\ \eta_i > 0. \quad (4)$$

$$a_0(s) = s^{k_0}\prod_{i=1}^{\ell_0}(s^2 + \nu_i^2), \quad (5)$$

$$\nu_i > 0, \quad (6)$$

$$\widehat{a}(s) = \prod_{i=1}^{k_+}(s - \rho_i), \quad \rho_i > 0 \quad (7)$$

Consequently, the degree, n, of the polynomial $a(s)$ is expressed as

$$n = k_+ + k_- + k_0 + 2(\ell_+ + \ell_- + \ell_0)$$

A. Necessary Conditions

The following implications are easily shown for a polynomial $a \in \mathbb{R}$[s]:

$$\begin{aligned}
\Sigma(a) &\subset \mathbb{R}_- \cup \mathbb{C}_{s-} = \mathbb{C}_- \\
&\Rightarrow \text{ all coefficients positive : } \quad a \in \mathbb{R}_+[s]. \\
\Sigma(a) &\subset \mathbb{R}_- \cup \mathbb{C}_{s-} \cup \{0\} \cup \mathbb{C}_{s0} \\
&= \mathbb{C}_- \cup \mathbb{C}_0 = \mathbb{C}_+^c \\
&\Rightarrow \text{ all coefficients nonnegative : } \quad a \in \overline{\mathbb{R}}_+[s].
\end{aligned}$$

III. PREPARATORY RESULT

The main theorem of this paper states that $r \in \mathcal{R}[x]$ iff $\Sigma(r) \subset \mathbb{C} \setminus \mathbb{R}_+$. Moreover, it follows from the proof of the main theorem that an auxiliary multiplier polynomial, q, may be found in $\mathbb{R}_+[x]$.

Let $a(s)$ be a polynomial with all its roots in $\mathbb{C}_{s+}$. We will show that a polynomial $q(s)$ exists such that all roots of the product $p(s) = a(s)q(s)$ has all positive coefficients, and that this is not possible if $a(s)$ has a root in $\mathbb{R}_+$. We shall call such a polynomial *complementary* to $a(s)$.
This implies that any polynomial with all positive coefficients has all its roots in $\mathbb{C}_s$, the complex plane with a slit.

Lemma 1: *The first degree polynomial, $(s-1)$, cannot be multiplied by a real polynomial to yield a polynomial with positive coefficients.*

Proof: Assume the contrary and let $q(s) = \sum_{i=0}^{n} q_i s^{n-i}$ be the multiplier polynomial. Then

$$(s-1)q(s) = q_0 s^{n+1} + \sum_{i=0}^{n-1} (q_{i+1} - q_i) s^i - q_n$$

With $q_0 = 1$, it is easily seen that the coefficients of the nonzero powers are strictly positive if

$$q_n > q_{n-1} > \cdots > q_1 > q_0 > 0$$

but then the constant term $-q_n$ is negative contradicting the assumption. □

Alternatively, as pointed out by a reviewer, the assumption $p(s) = (s-1)q(s) \in \mathbb{R}_+[s]$ yields the contradiction $0 = p(1) > 0$.

Lemma 2: *Given the second degree polynomial, $(s^2 - 2s\cos\alpha + 1)$, with $0 < \alpha \le \pi/2$, there exists an integer n such that the polynomial $c(s;\alpha,n) = (s^2 - 2s\cos\alpha + 1)(s+1)^n$ has positive coefficients. The smallest such n is upper bounded by $2\lfloor \frac{\cos\alpha}{1-\cos\alpha} \rfloor + 1$.*

Proof: The given polynomial, a, has the complex roots $r_1 = \cos\alpha + i\sin\alpha$, and $r_2 = \cos\alpha - i\sin\alpha$. The same holds for the polynomial c. In addition, $c(s)$ also has roots $r_3 = r_4 = \ldots = r_{n+2} = -1$. The coefficients of the polynomial $c(s) = \sum_{k=0}^{n+2} c_k s^{n+2-k}$ are given by Newton's identities:

$$\begin{aligned}
c_0 &= 1 \\
c_1 &= (-1)\sum_{i=1}^{n+2} r_i \\
c_2 &= (-1)^2 \sum_{i\neq j} r_i r_j \\
&\vdots \\
c_k &= (-1)^k \sum_{\substack{i_1,\ldots,i_k \\ \text{disjoint}}} \prod_{p=1}^{k} r_{i_p} \\
&\vdots \\
c_{n+2} &= (-1)^{n+2} \prod_{p=1}^{n+2} r_p .
\end{aligned}$$

In terms of the zero structure of $c(s)$, it is directly seen that

$$\begin{aligned}
c_1 &= (-1)\left[-n + 2\cos\alpha\right] \\
c_2 &= (-1)^2 \left[\binom{n}{2}(-1)^2 + \binom{n}{1}(-1)^1 2\cos\alpha + 1\right] \\
c_3 &= (-1)^3 \left[\binom{n}{3}(-1)^3 + \binom{n}{2}(-1)^2 2\cos\alpha + \binom{n}{1}(-1)\right] \\
c_4 &= (-1)^4 \left[\binom{n}{4}(-1)^4 + \binom{n}{3}(-1)^3 2\cos\alpha + \binom{n}{2}(-1)^2\right] \\
&\vdots \\
c_k &= (-1)^k \left[\binom{n}{k}(-1)^k + \binom{n}{k-1}(-1)^{k-1} 2\cos\alpha + \binom{n}{k-2}(-1)^{k-2}\right] \\
&\vdots \\
c_{n+1} &= (-1)^{n+1} \left[\binom{n}{n}(-1)^n 2\cos\alpha + \binom{n}{n-1}(-1)^{n-1}\right] \\
c_{n+2} &= (-1)^{n+2} \left[\binom{n}{n}(-1)^n\right].
\end{aligned}$$

This gives

$$\begin{aligned}
c_1 &= n - 2\cos\alpha \\
c_2 &= \binom{n}{2} - \binom{n}{1} 2\cos\alpha + 1 \\
c_3 &= \binom{n}{3} - \binom{n}{2} 2\cos\alpha + \binom{n}{2} \\
c_4 &= \binom{n}{4} - \binom{n}{3} 2\cos\alpha + \binom{n}{2} \\
&\vdots \\
c_k &= \binom{n}{k} - \binom{n}{k-1} 2\cos\alpha + \binom{n}{k-2} \\
&\vdots \\
c_{n+1} &= -\binom{n}{n} 2\cos\alpha + \binom{n}{n-1} \\
c_{n+2} &= 1
\end{aligned}$$

It follows that all coefficients are positive if $\cos\alpha < n/2$, and if for all $k = 2,\ldots,n$

$$\begin{aligned}
\cos\alpha &< \frac{\frac{1}{k!(n-k)!} + \frac{1}{(k-2)!(n-k+2)!}}{\frac{2}{(k-1)!(n-k+1)!}} \\
&= \frac{(n-k+2)(n-k+1) + k(k-1)}{2k(n-k+2)}.
\end{aligned}$$

Note the equalities $c_0 = c_{n+2}$, $c_1 = c_{n+1}$, and by symmetry properties of the binomial coefficients, $c_k = c_{n+2-k}$ in general. Consider now the cosine bounds

$$B(k,n) = \frac{(n-k+2)(n-k+1) + k(k-1)}{2k(n-k+2)}.$$

If n is even, set $n = 2m$, and let $k = m+1+p$ for some integer p. It follows that

$$\begin{aligned}
B(m+1+p, 2m) &= \frac{m(m+1) + p^2}{(m+1+p)(m-p+1)} \\
&= \frac{m(m+1) + p^2}{(m+1)^2 - p^2}.
\end{aligned}$$

It is now readily seen that this bound is minimal for $p = 0$, hence the minimum of $B(k, 2m)$ occurs for $k = m+1$, and equals $B(m+1, 2m) = \frac{m}{m+1}$.

Likewise, if n is odd, set $n = 2m+1$. The expression $k = m+1+p$ leads to

$$\begin{aligned} B(m+1+p, 2m+1) &= \frac{m^2+2m+1+p^2-p}{(m+1+p)(m+2-p)} \\ &= \frac{(m+1)^2+(p^2-p)}{(m+1)(m+2)-(p^2-p)}. \end{aligned}$$

This expression is minimal for $p^2 - p = 0$, which yields $p = 1$ and $p = 0$. Hence the minimal bound is obtained for $k = m+1$ and $k = m+2$, and evaluates to

$$B(m+1, 2m+1) = B(m+2, 2m+1) = \frac{m+1}{m+2}.$$

Observe that $\frac{m}{m+1} < \frac{m+1}{m+2}$ is true for all $m \in \mathbb{N}$ since $m(m+2) = (m+1)^2 - 1$. Let $U_k = \left[\frac{k}{k+1}, \frac{k+1}{k+2}\right)$, then the disjoint union $U = \bigcup_{k=0}^{\infty} U_k$ is equivalent to $[0, 1)$. The cos function restricted to the interval $\left(0, \frac{\pi}{2}\right]$ is a surjective mapping to $[0, 1)$, and so for any given $r \in U$ there exist $\alpha \in \left(0, \frac{\pi}{2}\right]$. Consequently, given α in the statement of the Lemma, there exist $M \in \mathbb{N}$ such that

$$\cos\alpha \in \left[\frac{M}{M+1}, \frac{M+1}{M+2}\right). \tag{8}$$

By the monotonicity, $\cos\alpha < \frac{m+1}{m+2}$ for all $m \geq M$. Therefore, for any $n \geq 2M+1$, $c(s)$ has positive coefficients. □

By (8), the left hand side inequality gives $M \leq \frac{\cos\alpha}{1-\cos\alpha}$, while the right hand side inequality yields $M > \frac{2\cos\alpha - 1}{1-\cos\alpha}$. This gives

$$\frac{\cos\alpha}{1-\cos\alpha} - 1 < M \leq \frac{\cos\alpha}{1-\cos\alpha}.$$

Since the size of the interval is 1 and the equality only holds for the right hand side, M is uniquely defined as $M = \lfloor \frac{\cos\alpha}{1-\cos\alpha} \rfloor$. The resulting bound on the degree is $N = 2M+1$, and is displayed in Figure 1 in function of the argument $\alpha \in (0, \frac{\pi}{2})$. From Figure 2 it can be seen that

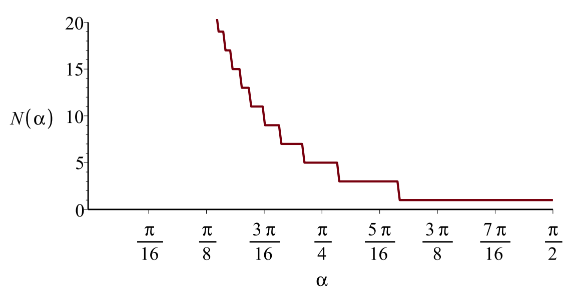

Fig. 1. The bound $N(\alpha)$ on the degree for the auxiliary polynomial.

the degree approaches $2/\alpha^2$ as $\alpha \to 0$.
For the quadratic polynomial $x^2 - 2\frac{p}{q}x + 1$ with $p, q \in \mathbb{Z}$ relatively prime and $0 < p < q$, we can apply the bound given by Zaïmi [15] for the corresponding problem in $\mathbb{Q}[x]$. Letting $\alpha = \frac{p}{q}$, the roots of the quadratic polynomial are $a_\pm = \cos\alpha \pm i\sin\alpha$, with $\sin\alpha \neq 0$. These are algebraic

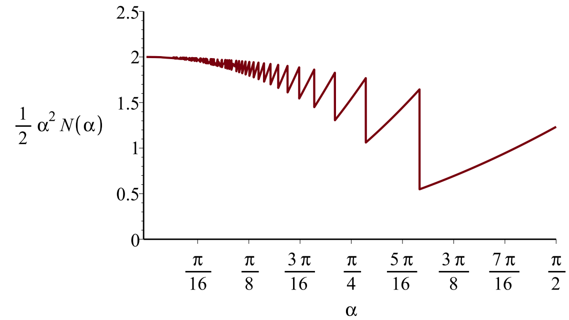

Fig. 2. Asymptotic behavior of $\frac{1}{2}\alpha^2 N(\alpha)$.

numbers with minimal polynomial $\mathrm{Min}_{a_\pm}(x) = x^2 - 2\frac{p}{q}x + 1$. Hence the degree is $d = 2$. The discriminant is

$$lcm(1, q)^{2d-2}(a_+ - a_-)^2 = -4(q^2 - p^2).$$

The Mahler measure of a_+ is

$$M = lcm(1, q)\max(1, |a_+|)\max(1, |a_-|) = q,$$

so that Zaïmi's upper bound for the degree is

$$\frac{2\pi d}{\arcsin(|\Delta|^{1/2} d^{-(d+3)/2} M^{-d+1})} = \frac{4\pi}{\arcsin(2^{-3/2}|\sqrt{1-\frac{p^2}{q^2}})}$$

Our bound gives

$$1 + 2\lfloor \frac{p}{q-p} \rfloor,$$

which is more tight for $\alpha = \arccos(p/q) > 0.11185$ (rad). For $\alpha = \pi/3$ corresponding to $p = 1$ and $q = 2$, the bounds are 3 (ours) and 41 (Zaïmi). The minimal degree of $q(s)$ is also discussed in Theorem 6 of [4]. For the quadratic polynomial $x^2 - 2x\cos\theta + 1$, this degree is found to be $\lfloor \frac{\pi}{\theta} \rfloor - 1$. For $p = 1$, and $q = 2$, this gives 3.

IV. MAIN RESULT

Now, we are in a position to state our main result:

Theorem 1: *If a polynomial has all strictly positive coefficients then it cannot have a root in* $\overline{\mathbb{R}_+}$.

Proof: Let n be the degree of the polynomial. For $n = 1$, the result is obvious. Now proceed by induction. Assume that the statement is true for the order n. Pick $q(s)$ a polynomial of degree $n+1$ with all positive coefficients, then there exist $p(s)$, a polynomial of degree n with all positive coefficients, such that $q(s) = sp(s) + q_0$ where $q_0 > 0$. By the assumption, $p(s)$ cannot have roots in the positive real axis (including 0). It follows that the roots of $q(s)$ must satisfy

$$sp(s) = s\prod_{i=1}^{n}(s - c_i) = -q_0 < 0.$$

Taking the angular part of this complex equation

$$\arg(s) + \sum_{i=1}^{n}\arg(s - c_i) = \pi. \tag{9}$$

If there were a root on the positive real axis, then $\arg s = 0$, and for each real root $c_i < 0$ of $p(s)$ it also holds that $\arg(s - c_i) = 0$. For every complex conjugate pair, the sum of the arguments modulo 2π is zero as well. This contradicts the (9). Hence $q(s)$ cannot have a root on the positive real axis.□

Undoubtedly, the classical control engineer will recognize the celebrated root locus principle in the above proof. A simple alternative proof is to take $c \geq 0$, and note that then $p(c) \geq p(0) > 0$, hence $p(c) \neq 0$.
It should also be obvious that the above implication cannot be reversed. A simple counterexample is the second order polynomial $s^2 - 2s + 2$. It has no roots on the positive real axis, but fails to have all positive coefficients.
However, the following will be shown:

Theorem 2: *If $a(s)$ has no roots on the positive real axis (including zero), $\overline{\mathbb{R}_+}$, then there always exist a polynomial $p(s)$ such that the product $a(s)p(s)$ has all positive coefficients.*

This fact is known since E. Meissner [12] and A. Durand (see [3], Théorème 2), but we present an alternate proof.

Proof: Pick $a(s)$ such that no roots are on the positive real axis (including zero), then $a(s) = a_+(s)a_-(s)a_0(s)$ as defined in Eqn 1 where $k_+, k_0 = 0$. By the construction of the factors, a_+ have all positive coefficients. It is enough to show that there exist $p(s)$ such that $a_-(s)a_0(s)p(s)$ have all positive coefficients. Since all the roots in $a_-(s)a_0(s)$ are complex conjugated, the polar expression of the roots can be written as $s_j = r_j e^{i\theta_j}$ and $s_j^* = r_j e^{-i\theta_j}$ where $r_j > 0$ and $\theta_j \in (0, \frac{\pi}{s}]$ for all $1 \leq j \leq n - k_- - 2\ell_-$. The multiplication of two conjugated polynomials can also be written in terms of r_j and θ_j by $(s - s_j)(s - s_j^*) = (s^2 - 2\cos\theta_j s + r_j^2)$. By factoring r_j^2, we have $((\frac{s}{r_j})^2 - 2\cos\theta_j \frac{s}{r_j} + 1)$. Let $s_1 = \frac{s}{r_j}$, then by Lemma 2, there exist n_j such that $(s_1^2 - 2\cos\theta_j s_1 + 1)(s_1 + 1)^{n_j}$ have all positive coefficients. Since r_j is a positive constant, replacing $s_1 = \frac{s}{r_j}$ would not change the sign of the coefficients. Finally, by letting $q(s) = \prod_{j=1}^{n-k_- - 2\ell_-} (\frac{s}{r_j} + 1)^{n_j}$ be a product of all such polynomials, $a_-(s)a_0(s)p(s)q(s)$ have all positive coefficients. □

Remark 1: The interesting fact of the theorem is that we place auxiliary roots with the same radius r_j in the negative real axis. The multiplicity of the roots solely depend on the angle θ_j. In contrast, the roots of the multiplier can be only computed numerically using Brunotte's algorithm [4]. We have tried to multiply with $(1 - 2\cos\beta s + s^2)$ instead of $(s + 1)^n$. It turns out that if $\alpha > \pi/4$ there always exist β which makes all the coefficient positive, as is easily seen, since $(1 - 2s\cos\alpha + s^2)(1 - 2s\cos\beta + s^2) = 1 - 2s(\cos\alpha + \cos\beta) + 2s^2(1 + \cos\alpha\cos\beta) - 2s^3(\cos\alpha + \cos\beta) + s^4$ leads to the conditions $\cos\alpha + \cos\beta < 0$ and $1 + 2\cos\alpha\cos\beta > 0$, and with $\cos\alpha > 0$ this yields

$$-\frac{1}{2\cos\alpha} < \cos\beta < -\cos\alpha.$$

Remark 2: For an analogue for a product with nonnegative coefficients, see [6], [10].

The foregoing results lead to an upper bound for the degree of the complementary polynomial.

Theorem 3: *Let the polynomial p be monic, then*

$$p \in \mathcal{R}[s] \iff p \in \mathcal{P}[s].$$

Proof:
$\Rightarrow$: By definition, there exists $q \in \mathbb{R}_+[s]$ such that $pq \in \mathbb{R}_+[s]$. Any polynomial in $\mathbb{R}[s]$ assumes strictly positive values on $\mathbb{R}_+$ as its evaluation is a sum of products of all strictly positive quantities. Hence $p(r)q(r)$ and $q(r)$ are strictly positive for all $r > 0$. Hence $p(r) > 0 \, \forall r \in \mathbb{R}$.
$\Leftarrow$: If $p(r) > 0$ for all $r > 0$, then p obviously cannot have a zero on $\mathbb{R}_+$. By the main theorem, this characterizes the set $\mathcal{R}[x]$. □

Note that Theorem 3 gives an alternate characterization $\mathcal{P}[s] = \mathcal{R}[s]$. In fact a little more can be said by invoking the continuity of polynomials:

$$p \in \mathcal{R}[s] \Leftrightarrow \Re p(s) > 0 \quad s \in \mathcal{N}_+$$

where $\mathcal{N}_+$ is a sufficiently small neighborhood of the positive real axis. With this, the notion of a *positive real function* comes to mind. The function $f(s)$ is positive real if $\Re f(s) > 0$ whenever $\Re s > 0$ and $f(r) > 0$ for $r \in \mathbb{R}_+$.

V. Topology

The following are results of a more topological nature.

Lemma 3: *If $p \in \mathcal{P}[s]$, then $\Re p(s) > 0$ in a neighborhood of $\mathbb{R}_+$.*
Proof: Suppose there exist $\epsilon, r_1 > 0$ such that if $s \in \mathbb{C}$ satisfies $|s - r_1| < \epsilon$ and $\Im s \neq 0$, then $\Re p(s) = \alpha \leq 0$. Let $r_2 = p(r_1) > 0$, then by the continuity of p at r_1, there exist $s_1 \in \mathbb{C} \backslash \mathbb{R}_+$ such that $|s_1 - r_1| < \epsilon$ and $|p(s_1) - r_2| < r_2/2$. Let $p(s_1) := \alpha_1 + \beta_1 i$, then $\alpha \leq 0$, and so,

$$|p(s_1) - r_2| = \sqrt{(\alpha_1 - r_2)^2 + \beta_1^2} \geq \sqrt{r_2^2 + \beta_1^2} \geq r_2 \quad (10)$$

This is a contradiction, hence, the lemma holds. □

Theorem 4: *Let $p \in \mathcal{R}[s]$, then there exists a positive real $\gamma \in (0, 1)$ such that $P(s) = p(s^\gamma)$ is a positive real function.*

Proof: Since $\Re p(s)$ is positive over the wedge $|\arg s| < \theta$ for some $\theta \in (0, \pi/2)$, the result follows by applying the conformal map $s \to s^\gamma$ with $\gamma = \frac{2\theta}{\pi}$. □

VI. An Alternative Solution

Consider a pair of complex conjugate poles, $s_\pm = \mathrm{e}^{\pm i\pi/N}$, where N is an odd integer. It is easily verified that $s_\pm$ are N-th roots of -1. All N-th roots are of the form $\mathrm{e}^{i(2k+1)\pi/N}$, where $k = 0,\ldots,N-1$. Indeed, $[\mathrm{e}^{i(2k+1)\pi/N}]^N = \mathrm{e}^{i(2k+1)\pi} = -1$. Consequently, the product of the factors $(s - \mathrm{e}^{i(2k+1)\pi/N})$ is an N-th order polynomial, $s^N + 1$. Thus we conclude that a complementary polynomial for $s^2 - 2\cos\frac{\pi}{N}s + 1$ is the polynomial

$$q_0(s) = (s+1)\prod_{k=1}^{N-2}\left(s^2 - 2\cos\frac{(2k+1)\pi}{N} + 1\right).$$

The problem is that this only yields a product with nonnegative coefficients. But any polynomial of the form s^N+1 can be multiplied by an arbitrary monic polynomial, q_1 in $\mathcal{R}[s]$ of degree $N-1$ to yield a polynomial in $\mathcal{R}[s]$ of degree $2N-1$. Hence the full complementary polynomial is the polynomial $q_0(s)q_1(s)$. Using our main result, the degree would have been equal to $2\left\lfloor\frac{\cos\frac{\pi}{N}}{1-\cos\frac{\pi}{N}}\right\rfloor + 1$.
While the degree using the alternative method increases linearly with N, as opposed to quadratically using our main result, it turns out that the result is very sensitive towards perturbations, as shown below.

A. Perturbation analysis

Let the given complex conjugate pole pair be $\mathrm{e}^{i(\epsilon+\pi/N)}$. Then the product is

$$(s^N+1)\frac{s^2 - 2\cos(\pi/N+\epsilon)s + 1}{s^2 - 2\cos(\pi/N)s + 1}.$$

This is

$$(s^N{+}1)\left[1 - 2s\frac{\cos(\pi/N)(1-\cos\epsilon) + \sin(\pi/N)\sin\epsilon}{s^2 - 2\cos(\pi/N)s + 1}\right]$$

or

$$(s^N{+}1)\left[\frac{(s^2+1)(1-\cos\epsilon) + 2\sin(\frac{\pi}{N})\sin\epsilon}{s^2 - 2\cos(\frac{\pi}{N})s + 1} + \cos(\frac{\pi}{N})\right].$$

which also reduces to

$$(s^N{+}1)\Big[2\sin(\frac{\epsilon}{2})\frac{(s^2+1)(\sin\epsilon/2) + 2\sin(\pi/N)\cos(\epsilon/2)}{s^2 - 2\cos(\pi/N)s + 1} + \cos(\pi/N)\Big].$$

which for small $|\epsilon|$ is approximately

$$(s^N+1)\left[2\epsilon\frac{(s^2+1)\sin(\pi/N)}{s^2 - 2\cos(\pi/N)s + 1} + \cos(\pi/N)\right].$$

B. Generalization

Suppose now that the complex conjugate pair of roots is $s_\pm = \mathrm{e}^{i(2k+1)\pi/N}$ for some k such that $\cos((2k+1)\pi/N) > 0$. Then it is necessary that $2k+1 < N/2$. Thus, $k < \frac{N-2}{4}$.

VII. Application: Positive system realization

Our main result leads now to the following application to realizability of systems by a positive system: Consider an irreducible proper transfer function, $H(s) = \frac{b(s)}{a(s)}$, having neither poles nor zeros on the positive real axis, $\overline{\mathbb{R}_+}$. In fact, irreducibility may be relaxed to "numerator, $b(s)$, and denominator, $a(s)$, not having common roots on the positive real axis". We shall describe the latter by $\overline{\mathbb{R}}_+$-irreducibility. We get the following theorem:

Theorem 5: *A proper $\overline{\mathbb{R}}_+$-irreducible rational function $H(s) = \frac{b(s)}{a(s)}$ having neither poles nor zeros on the positive real axis is always realizable in a form using only strictly positive multipliers (except, of course, for the standard comparator at the feedback).*

Proof: By Theorem 2, auxiliary polynomials $\alpha(s)$ and $\beta(s)$ exist, both in $\mathbb{R}_+[s]$, such that $A(s) = a(s)\alpha(s)$ and $B(s) = b(s)\beta(s)$ are in $\mathbb{R}_+[s]$.
Let now

$$\frac{b(s)}{a(s)}\cdot\frac{\beta(s)}{\alpha(s)} = \frac{B(s)}{A(s)} = H_s(s),$$

with $H_s(s) \in \mathbb{R}_+(s)$, the set of rational functions in s with strictly positive coefficients. But then the product (before reduction!)

$$H_s(s)\cdot\frac{\alpha(s)}{\beta(s)} \stackrel{\text{def}}{=} H_s(s)H_a(s),$$

where we defined an *auxiliary transfer function*, $H_a(s) \in \mathbb{R}_+(s)$, also has all positive coefficients. Its denominator has degree $\delta_a + \delta_\alpha + \delta_\beta$, and its *reduced* transfer function is the given proper $H(s)$. Now the product H_sH_a is realizable without differentiators (for continuous time systems) if $\delta_a + \delta_\alpha + \delta_\beta \geq \delta_b + \delta_\beta + \delta_\alpha$, which holds since $H(s)$ was assumed to be proper, thus proving the theorem. $\square$

VIII. Conclusions and Beyond

We characterized the set of polynomials with positive coefficients in terms of their root locations. The important root set is thus the slitted plane $\mathbb{C}_s = \mathbb{C}\setminus\mathbb{R}_+$. What can be said if instead we take for the root set the rotated slitted plane $\mathrm{e}^{i\theta}\mathbb{C}_s$? (Ans: A scaled version). In view of [4] the polynomial $q(s) = (s+1)^{N(\alpha)}$ is not necessarily the complementary polynomial of $r(s) = s^2 - 2s\cos\alpha + 1$ with lowest degree, but it is readily computable (its coefficients are the binomial coefficients). We also made a connection to the notion of positive realness, and applied our results to the realization of positive systems.

Acknowledgment

The authors thank the anonymous reviewers for their helpful comments.

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