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Convergent collocation methods for Hamilton-Jacobi-Bellman equations*

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Abstract— **We report a convergence result of kernel-based collocation methods for Hamilton-Jacobi-Bellman equations. In particular, we find the class of kernels and the structure of collocation points explicitly under which the process of iterative interpolation is stable, and show that the functions constructed by the collocation methods converge to a unique viscosity solution of the Hamilton-Jacobi-Bellman equations.**

I. INTRODUCTION

In this talk, we are concerned with the numerical methods for Hamilton-Jacobi-Bellman equations defined on $[0,T]\times\mathbb{R}^d$:

$$\begin{cases} \partial_t v + \inf_{a\in U} H^a(t,x,v(t,x),Dv(t,x),D^2v(t,x)) = 0, \\ v(T,x) = f(x), \end{cases} \tag{1}$$

where H^a denotes the Hamiltonian for a control $a\in U$ arising from a stochastic control problem over finite horizon T. Here we have denoted by ∂_t the partial differential with respect to the time variable t, by D^1 and D^2 the gradient and Hessian with respect to the spacial variable x, respectively. The conditions imposed on H^a, A, and f are described in Section II below.

Several numerical methods for fully nonlinear parabolic partial differential equations (1) are proposed. For example, the finite difference methods (see, e.g., Kushner and Dupuis [1] and Bonnans and Zidani [2]), the finite-element like methods (see, e.g., Camilli and Falcone [3] and Debrabant and Jakobsen [4]), and the probabilistic methods (see, e.g., Pagès et al. [5], Fahim et al. [6] Guo et.al [7] and Nakano [8]).

An another possible approach to (1) is to use the kernel-based (meshfree) collocation method proposed by Kansa [9]. In this method, we seek an approximate solution of the form of a linear combination of a radial basis function (e.g., multiquadrics in the Kansa's original work). Substituting this form into a partial differential equation leads to an equation for the collocation points. Then the approximate solution is constructed by the kernel-based interpolation of these collocation points. As for rigorous convergence, Schaback [10] and Nakano [11] study the case of linear operator equations and fully nonlinear parabolic equations, respectively. However, at least for parabolic equations, there is little known about explicit examples of the grid structure and kernel functions that ensure rigorous convergence. An exception is Hon et.al [12], where an error bound is obtained for a special heat equation in one dimension. In this report, we show that the functions constructed by the collocation methods converge to a unique viscosity solution of (1) under the same condition imposed in [13], where the convergence of the collocation method for Zakai equations is analyzed.

II. KERNEL-BASED COLLOCATION METHODS

We consider the terminal value problem (1) under the following assumptions:

Assumption 2.1: There exists a positive constant C_0 such that the following are satisfied:

(i) U is a subset of some Euclidean space.

(ii) For $a\in U$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $z\in\mathbb{R}$, $p\in\mathbb{R}^d$, and $\gamma,\gamma'\in\mathbb{S}^d$ with $\gamma\ge\gamma'$,

$$H^a(t,x,z,p,\gamma)\le H^a(t,x,z,p,\gamma').$$

(iii) There exist a continuous function F_0 on $[0,T]$ such that

$$\begin{aligned}&|H^a(t,x,z,p,\gamma)-H^a(t',x',z',p',\gamma')|\\ &\le |F_0(t)-F_0(t')|\\ &\quad + C_0(|x-x'|+|z-z'|+|p-p'|+|\gamma-\gamma'|)\end{aligned}$$

for $a\in U$, $t,t'\in[0,T]$, $x,x'\in\mathbb{R}^d$, $z,z'\in\mathbb{R}$, $p,p'\in\mathbb{R}^d$, and $\gamma,\gamma'\in\mathbb{S}^d$.

(iv) For $a\in U$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $z\in\mathbb{R}$, $p\in\mathbb{R}^d$, and $\gamma\in\mathbb{S}^d$,

$$|H^a(t,x,z,p,\gamma)|\le C_0(1+|z|+|p|+|\gamma|)$$

(v) The function f is continuous and bounded on $\mathbb{R}^d$.

We assume that the following comparison principle holds:

Assumption 2.2: For every bounded, upper-semicontinuous viscosity subsolution u of (1) and bounded lower-semicontinuous viscosity supersolution w of (1), we have

$$u(t,x)\le w(t,x),\quad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Under Assumptions 2.1 and 2.2, there exists a unique continuous viscosity solution v of (1). See [14].

We shall recall from [11] the kernel-based collocation methods with time discretization for (1). In what follows, the function Φ is assumed to be the Wendland kernel $\Phi(x)=\phi_{d,\tau}(|x|)$ with fixed $\tau\in\mathbb{N}$, which is given by

$$\phi_{d,\tau}(r)=\begin{cases}\displaystyle\int_r^1 s(1-s)^{\ell}(s^2-r^2)^{\tau-1}ds, & 0\le r\le 1,\\ 0, & r>1,\end{cases}$$

where $\ell=\max\{k\in\mathbb{Z}: k\le d/2\}+\tau+1$. Let $h>0$ be a parameter that describes approximate solutions, $\Gamma=$

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$\{x^{(1)}, \dots, x^{(N)}\} \subset (-R, R)^d$ with $R > 1$, and $\{t_0, \dots, t_n\}$ the set of time grid points such that $0 = t_0 < t_1 < \cdots < t_n = T$. Then set $A = \{\Phi(x^{(i)} - x^{(j)})\}_{1 \le i,j \le N}$ and think of the interpolant

$$v^h(t_k, x) = \sum_{j=1}^{N} (A^{-1} v_k^h)_j \Phi(x - x^{(j)}), \quad x \in \mathbb{R}^d, \qquad (2)$$

of $v_k^h = (v_{k,1}^h, \dots, v_{k,N}^h)^\mathsf{T} \in \mathbb{R}^N$ to be specified below. Substituting this into the time discretized equation

$$\frac{v(t_{k+1}, x) - v(t_k, x)}{t_{k+1} - t_k} + H(t_{k+1}, x; v(t_{k+1}, \cdot)) \simeq 0,$$

we derive the following equation for $\{v_k^h\}$:

$$v_{k+1,j}^h - v_{k,j}^h = -(t_{k+1} - t_k) H_{k+1,j}(v_{k+1}^h), \quad k = 0, \dots, n-1, \;\; j = 1, \dots, N.$$

Here, for any C^2-function φ on $\mathbb{R}^d$,

$$H(t, x; \varphi) = \inf_{a \in U} H^a(t, x, \varphi(x), D\varphi(x), D^2\varphi(x)), \quad x \in \mathbb{R}^d,$$

and $H_{k,j}(v_k^h) = H(t_k, x^{(j)}; v^h(t_k, \cdot))$. The terminal condition leads to $v_{n,j}^h = f(x^{(j)})$, $j = 1, \dots, N$. Thus, denoting $H_k(v_k^h) = (H_{k,1}(v_k^h), \dots, H_{k,N}(v_k^h))^\mathsf{T}$, we get

$$\begin{cases} v_k^h = v_{k+1}^h + (t_{k+1} - t_i) H_{k+1}(v_{k+1}^h), & k = 0, \dots, n-1, \\ v_n^h = f|_\Gamma. \end{cases} \qquad (3)$$

Consequently, we define the function $v^h(t_k, x)$, a candidate of an approximate solution of (1), by (2) with $\{v_k^h\}$ determined by the equation (3).

III. A CONVERGENCE RESULT

To discuss the error of the approximation above, set $\Delta t = \max_{1 \le i \le n}(t_i - t_{i-1})$ and consider the Hausdorff distance $\Delta_1 x$ between Γ and $(-R, R)^d$, and the separation distance $\Delta_2 x$ defined respectively by

$$\Delta_1 x = \sup_{x \in (-R,R)^d} \min_{j=1,\dots,N} |x - x^{(j)}|,$$
$$\Delta_2 x = \frac{1}{2} \min_{i \ne j} |x^{(i)} - x^{(j)}|.$$

Then suppose that Δt, R, N, $\Delta_1 x$ and $\Delta_2 x$ are functions of h. In what follows, $\#\mathcal{K}$ denotes the cardinality of a finite set $\mathcal{K}$.

Assumption 3.1: (i) The parameters Δt, R, N, and $\Delta_1 x$ satisfy $\Delta t \to 0$, $R \to \infty$, $N \to \infty$, and $\Delta_1 x \to 0$ as $h \searrow 0$.

(ii) There exist c_1, c_2, c_3, c_4 and λ, positive constants independent of h, such that for any $i = 1, \dots, N$,

$$\# \left\{ j \in \{1, \dots, N\} : |(A^{-1})_{ij}| > c_1 \frac{(\Delta_2 x)^d}{N} \right\} \le c_2 (\Delta_2 x)^{-\lambda d},$$

and that

$$c_3 (\Delta_2 x)^{-(1+\lambda)d} \le R^{1/2} \le c_4 (\Delta_1 x)^{-(\tau - 3/2)/d}.$$

Remark 3.2: In the case of a uniform grid, a sufficient condition for which the latter part of Assumption 3.1 (ii) holds is

$$c_5 N^{(1 - 1/(1+2d(1+\lambda))) \frac{1}{d}} \le R \le c_6 N^{(1 - d/(d + 2\tau - 3)) \frac{1}{d}}$$

with $\tau \ge 3/2 + (1+\lambda) d^2$, for some positive constants c_5 and c_6.

The following result tells us that the process of iterative kernel-based interpolation is stable, which is a key to our convergence analysis. For a proof we refer to [13].

Lemma 3.3: Suppose that Assumption 3.1 and $\tau \ge 3$ hold. Then, there exists $h_0 > 0$ such that for $|\alpha|_1 \le 3$,

$$\sup_{0 < h \le h_0} \sup_{x \in (-R,R)^d} |D^\alpha I(g)(x)| \le C \max_{j=1,\dots,N} |g(x^{(j)})|.$$

Here is our main result.

Theorem 3.4: Suppose that Assumptions 2.1, 2.2 and 3.1 hold. Suppose moreover that $\tau \ge 3$. Then we have

$$\lim_{t_k \to t,\, h \searrow 0} v^h(t_k, x) = v(t, x),$$

uniformly on any compact subset of $\mathbb{R}^d$.

Remark 3.5: In [11], the convergence result as in Theorem 3.4 is proved under more normative assumptions. Here, we reveal explicit conditions under which the convergence is guaranteed.

We refer to [15] for details.

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