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Further Study on Parameter Bound of Surplus-Based Averaging Algorithm

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***Abstract*— We study a continuous-time surplus-based algorithm for multi-agent average consensus, and derive a tight upper bound on the key parameter included in this algorithm that ensures convergence over strongly connected and balanced digraphs. Based on this result, it is suggested through extensive simulation that, for the same number of agents, the upper bound for cyclic digraphs be smaller than that for other strongly connected and possibly unbalanced digraphs; this implies that as long as the parameter satisfies the upper bound for cyclic digraphs, this parameter can work for other digraphs.**

I. INTRODUCTION

The problem of multi-agent average consensus has attracted significant attention in the systems and control community [1]–[3]. In this problem, a network of interconnected agents aims to reach an agreement on the average value of their initial states, through only local communications among neighbors. The inter-agent communication topology is modeled by a directed graph (or digraph).

In [2] a basic consensus algorithm was introduced but the consensus value depends on the communication topology and the agents' initial states. By contrast, in [3] a discrete-time "surplus-based" algorithm was proposed which provably achieves average consensus on arbitrary strongly connected digraphs. In this algorithm, a variable called "surplus" is augmented to each agent, which tracks the state changes of the associated agent; collectively these surplus variables keep the information of state sum shift, thereby achieving average consensus even if the digraph is not *balanced* (the latter requires that each agent maintains the same amount of incoming and outgoing information).

Despite many developments of the surplus-based algorithm [3]–[6], a fundamental issue remains unsolved. There is a parameter in the algorithm, whose magnitude must be "sufficiently small" in order to ensure the convergence of the algorithm. An explicit and tight bound on the parameter is yet unknown. For the discrete-time surplus-based algorithm, [3] presented an explicit, but highly conservative upper bound based on a matrix perturbation result. Even for several special digraphs (balanced, cyclic, and undirected), the bounds reported in [3] are not tight. For the continuous-time version in [6], there has not been any bound result reported.

In this paper, we study upper bounds for the parameter in the continuous-time surplus-based algorithm. In order to derive tight bounds, we focus on balanced digraphs and directly analyze the eigenvalues of the multi-agent system as functions of the parameter. Our contribution is that we derive a tight upper bound for the parameter for strongly connected and balanced digraphs, which is explicit in terms of the real and imaginary parts of the graph Laplacian's eigenvalues.

In addition, we show by extensive simulation that, for the same number of agents, the bound for cyclic digraphs is plausibly smaller than that for any other strongly connected and unbalanced digraphs; this means that as long as the parameter satisfies the (tight) bound for cyclic digraphs, the surplus-based algorithm with this parameter value may work for other possibly unbalanced digraphs.

The rest of this paper is organized as follows. Section II introduces communication digraphs and the continuous-time surplus-based average consensus algorithm. Section III presents our main result; a tight upper bound on the parameter is derived for balanced digraphs. Section IV provides illustrative simulation examples, and finally Section V states our conclusions.

II. PRELIMINARIES

In this paper, we will use the following notation. Denote $\mathbb{R}_+$ as the set of positive real numbers, I_n the $n \times n$ identity matrix, O_n the $n \times n$ zero matrix, and j (roman) the imaginary unit, i.e. $\mathrm{j} := \sqrt{-1}$. Also define $\mathbf{0}_n := [0 \cdots 0]^\top \in \mathbb{R}^n$ and $\mathbf{1}_n := [1 \cdots 1]^\top \in \mathbb{R}^n$.

A. Communication graphs

Given a multi-agent system, we represent the multi-agent system and the interactions between the agents by a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with node set $\mathcal{V} = \{v_1, \ldots, v_n\}$ and directed edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The graph is assumed to be directed, unweighted, and time-invariant. The node $v_i \in \mathcal{V}$ represents the ith agent. The ith agent receives information from the jth agent if and only if $(v_j, v_i) \in \mathcal{E}$. We define the set of "in-neighbors" of the ith agent by $\mathcal{N}_i^{\mathrm{I}} := \{v_k | (v_k, v_i) \in \mathcal{E}\}$ and the set of "out-neighbors" by $\mathcal{N}_i^{\mathrm{O}} := \{v_k | (v_i, v_k) \in \mathcal{E}\}$. For $i, j = 1, \ldots, n$, define receiving weight a_{ij} and sending weight b_{ij} by

$$a_{ij} := \begin{cases} 1, & v_j \in \mathcal{N}_i^{\mathrm{I}} \\ 0, & v_j \notin \mathcal{N}_i^{\mathrm{I}} \end{cases}, \quad b_{ij} := \begin{cases} 1, & v_j \in \mathcal{N}_i^{\mathrm{O}} \\ 0, & v_j \notin \mathcal{N}_i^{\mathrm{O}} \end{cases}$$

respectively. It is evident that $a_{ij} = b_{ji}$ for $i, j = 1, \ldots, n$.

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The *in-Laplacian matrix* $L_{\rm I} := [l_{ij}^{\rm I}] \in \mathbb{R}^{n\times n}$ and the *out-Laplacian matrix* $L_{\rm O} := [l_{ij}^{\rm O}] \in \mathbb{R}^{n\times n}$ are defined according to

$$l_{ij}^{\rm I} := \begin{cases} \sum_{k=1}^{n} a_{ik}, & j=i \\ -a_{ij}, & j\neq i \end{cases}, \quad l_{ij}^{\rm O} := \begin{cases} \sum_{k=1}^{n} b_{ik}, & j=i \\ -b_{ij}, & j\neq i \end{cases}$$

respectively. We say that there exists a *directed path* from $v_j \in \mathcal{V}$ to $v_i \in \mathcal{V}$ if there are directed edges from v_j to v_i, i.e.

$$\left\{(v_j, v_{m_1}), (v_{m_1}, v_{m_2}), \ldots, (v_{m_{q-1}}, v_{m_q}), (v_{m_q}, v_i)\right\} \subset \mathcal{E}$$

for all $1 \le m_1, \ldots, m_q \le n$.

Below are two graphical conditions we shall use later.

Definition 1 The digraph $\mathcal{G}$ is *strongly connected* if there exists a directed path from any v_i to any v_j $(i, j = 1, \ldots, n, i \neq j)$.

Definition 2 The digraph $\mathcal{G}$ is said to be *balanced* if $l_{ii}^{\rm I} = l_{ii}^{\rm O}$ for every $i = 1, \cdots, n$, i.e. the number of in-neighbors of each agent and the number of its out-neighbors are the same.

Note that if $\mathcal{G}$ is the balanced digraph, then $L_{\rm I} = L_{\rm O}$, $L_{\rm I}\mathbf{1}_n = \mathbf{0}_n$ and $\mathbf{1}_n^\top L_{\rm I} = \mathbf{0}_n^\top$.

B. Average consensus by surplus-based algorithm

Let $x_i(t) \in \mathbb{R}$ $(i = 1, \ldots, n)$ be the state variable of the ith agent, and $x_i(0) = x_{i0}$ the initial value. The average consensus problem requires the state of each agent to converge to the average of the initial values, i.e. $x_i(t) \to (1/n)\sum_{i=1}^{n} x_{i0}$ as $t \to \infty$ for all i.

In the case where a digraph is balanced, the standard algorithm (cf. [2])

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}\left(x_j(t) - x_i(t)\right)$$

is available for the average consensus problem.

To achieve average consensus on arbitrary strongly connected digraphs (not necessarily *balanced*), [3] proposed a discrete-time surplus-based algorithm. The continuous-time counterpart is as follows:

$$\begin{aligned} \dot{x}_i(t) &= \sum_{j=1}^{n} a_{ij}\left(x_j(t) - x_i(t)\right) + \epsilon s_i(t) \\ \dot{s}_i(t) &= -\sum_{j=1}^{n} a_{ij}\left(x_j(t) - x_i(t)\right) - \epsilon s_i(t) \\ &\quad + \sum_{j=1}^{n} \left(b_{ji} s_j(t) - b_{ij} s_i(t)\right) \\ x_i(0) &= x_{i0} \in \mathbb{R},\ s_i(0) = 0 \end{aligned} \tag{1}$$

where $s_i(t) \in \mathbb{R}$ is called the surplus variable of the ith agent and $\epsilon > 0$ is a parameter which specifies the amount of surplus used in the x_i update. To write (1) in matrix form, define $x(t) := [x_1(t) \cdots x_n(t)]^\top$, $x_0 := [x_{10} \cdots x_{n0}]^\top$, $s(t) := [s_1(t) \cdots s_n(t)]^\top$, and hence

$$\begin{bmatrix} \dot{x}(t) \\ \dot{s}(t) \end{bmatrix} = M \begin{bmatrix} x(t) \\ s(t) \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \mathbf{0}_n \end{bmatrix}$$
$$\text{where } M = \begin{bmatrix} -L_{\rm I} & \epsilon I_n \\ L_{\rm I} & -L_{\rm O}^\top - \epsilon I_n \end{bmatrix} \in \mathbb{R}^{2n\times 2n}. \tag{2}$$

Observe that the column sums of matrix M are zero; thus $\dot{x}(t) + \dot{s}(t) = 0$, i.e. $x(t) + s(t)$ is a constant. By this algorithm, the multi-agent system achieves average consensus if the matrix M has a simple eigenvalue 0 and all the other eigenvalues have negative real parts. In particular, the following holds (cf. [3, Proposition 6]):

$$\begin{aligned} &\lim_{t\to\infty} \begin{bmatrix} x(t) \\ s(t) \end{bmatrix} = \frac{\mathbf{1_n}^\top x_0}{n} \begin{bmatrix} \mathbf{1}_n \\ \mathbf{0}_n \end{bmatrix} \\ &\Leftrightarrow 0 = \lambda_1(M) > \mathrm{Re}\{\lambda_2(M)\} \ge \cdots \ge \mathrm{Re}\{\lambda_{2n}(M)\}. \end{aligned}$$

The convergence result of the surplus-based algorithm is the following (cf. [3, Theorem 4]).

Lemma 1 Consider the algorithm (2) and suppose that the digraph $\mathcal{G}$ is strongly connected. If the parameter $\epsilon > 0$ is sufficient small, then the agents achieve average consensus.

By Lemma 1, we know that there always exists a positive ϵ that ensures convergence to average consensus. For the discrete-time surplus-based algorithm, [3] provided several bound results for ϵ, but even for special topologies such as balanced, cyclic, and undirected, the provided bounds are conservative. Moreover, there is no known bound result for ϵ for the continuous-time algorithm (2). We fill this gap in this paper, by deriving a tight upper bound on ϵ that guarantees average consensus over balanced digraphs using algorithm (2).

III. MAIN RESULT

Our main result states a tight upper bound on ϵ for ensuring convergence of algorithm (2) in the case of strongly connected and balanced digraphs.

Theorem 1 Consider the algorithm (2) with the positive parameter ϵ, and suppose that $\mathcal{G}$ is a strongly connected and balanced digraph. Let $\mu_i = p_i + \mathrm{j}q_i$ $(i = 1, \ldots, n)$ be the eigenvalues of $L_{\rm I}$. Then

(i) when there exists at least one eigenvalue of $L_{\rm I}$ such that $p_i < |q_i|$, the agents achieve average consensus if and only if the parameter ϵ in M satisfies the following inequality

$$0 < \epsilon < \min_{i\in S} \frac{2p_i^2}{|q_i| - p_i} \tag{3}$$

where $S := \{k \in \{1, \ldots, n\} \mid p_k < |q_k|\}$;

(ii) when $p_i \ge |q_i|$ for all $i = 1, \ldots, n$, i.e. S is empty, the agents achieve average consensus if and only if $\epsilon > 0$.

Theorem 1 states a tight upper bound on ϵ for algorithm (2) to achieve average consensus on strongly connected and balanced digraphs. This bound is in fact needed only when $p_i < |q_i|$ holds for some eigenvalue of the in-Laplacian

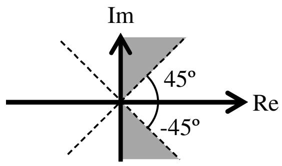

Fig. 1. Illustration of condition in Theorem 1

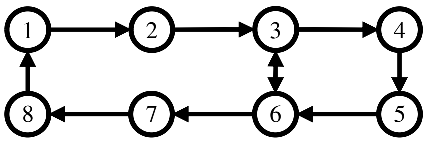

Fig. 2. A balanced digraph

matrix L_{I}, i.e. that eigenvalue lies in the shaded area in Fig. 1.

Compared to the counterpart result in [3] for the discrete-time algorithm on balanced digraphs, the bound in Theorem 1 is tight. This is because this bound is proved by directly analyzing the eigenvalues of the matrix M, whereas in [3], certain approximation methods were used. Moreover, case (ii) of Theorem 1 where any positive ϵ works is unique for the continuous-time algorithm; this is not the case for the discrete-time algorithm.

As a special case of strongly connected and balanced digraphs, the following result is derived for cyclic digraphs [7]. An (unweighted) cyclic digraph is one whose node set is $\mathcal{V} = \{v_1, \ldots, v_n\}$ and edge set is $\mathcal{E} = \{(1,2), (2,3), \ldots, (n-1, n), (n,1)\}$.

Corollary 1 Consider algorithm (2) with the positive parameter ϵ, and suppose that $\mathcal{G}$ is a cyclic digraph. Then

(i) for $2 \leq n \leq 4$, the agents achieve average consensus if and only if $\epsilon > 0$;

(ii) for $n \geq 5$, the agents achieve average consensus if and only if the parameter ϵ in M satisfies the following inequality

$$0 < \epsilon < \frac{4}{\tan^3 \theta_n - \tan^2 \theta_n + \tan \theta_n - 1} \tag{4}$$

where

$$\theta_n = \left(\frac{1}{2} - \frac{1}{n}\right).$$

IV. NUMERICAL EXAMPLES

We illustrate by simulations the derived tight bound for balanced strongly connected digraphs, and then we also consider unbalanced strongly connected digraphs using the above result.

A. Balanced digraph

As an exapmle of a balanced digraph, we use the digraph in Fig. 2 consisting of 8 nodes. The eigenvalues of L_{I} in algorithm (2) associated with this digraph are

$$0,\ 0.533 \pm 0.659\mathrm{j},\ 1.157 \pm 0.849\mathrm{j},\ 1.744 \pm 0.643\mathrm{j},\ 3.131$$

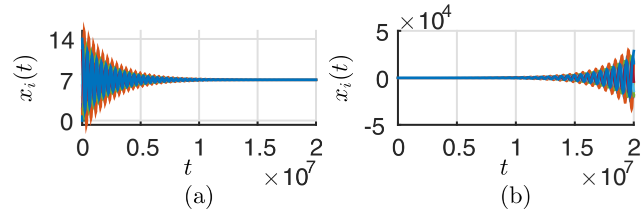

Fig. 3. Convergence results on the balanced digraph (Fig. 2): (a) $\epsilon = \bar{\epsilon} - 0.0001$ and (b) $\epsilon = \bar{\epsilon} + 0.0001$

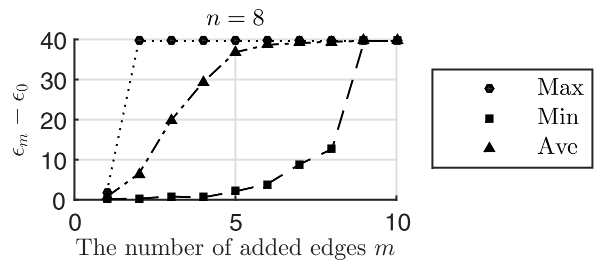

Fig. 4. Upper bound comparison between unbalanced digraphs and cyclic digraphs ($n = 8$)

and there exist eigenvalues satisfying $p_i < |q_i|$; in particular $S = \{0.533 \pm 0.659\mathrm{j}\}$. Then, according to Theorem 1, the upper bound of ϵ is $\bar{\epsilon} = 4.5231$.

We show the convergence result with $\epsilon = \bar{\epsilon} - 0.0001 = 4.5230$ and $\epsilon = \bar{\epsilon} + 0.0001 = 4.5232$ in Fig. 3 (a) and (b), respectively. The initial states of each agents are $x_{i0} = 2(i-1), i = 1, \ldots, 8$. When $\epsilon = 4.5230$, the states of all agents approach to the average of the initial states which is $\mathbf{1_n}^\top x_0 = 7$, but they diverge when $\epsilon = 4.5232$. This demonstrates the tightness of the bound we derived for ϵ.

B. Unbalanced strongly connected digraphs

Consider a cyclic digraph of 8 nodes, and we now randomly add some directed edges. There are in total 48 ($= 8 \times 7 - 8$) directed edges that are candidates to be added. From these 48 candidates, select $m \in \{1, \ldots, 48\}$ edges *uniformly* at random and add them to the cyclic digraph. Then numerically compute the upper bound ϵ_m, which is the maximal value of ϵ such that the maximum real part ρ of nonzero eigenvalues of M (recall that M has one zero eigenvalue) satisfies $\rho < 0$. If $\epsilon_m > 40$ then set $\epsilon_m = 40$, because this value is already much larger than $0.4142 (=: \epsilon_0)$ computed according to Corollary 1 as the upper bound for a cyclic digraph of 8 nodes. Repeat the above 200 times for the same m, and calculate ϵ_{max} as the maximum of ϵ_m, ϵ_{min} as the minimum and ϵ_{ave} as the average for each m.

In Fig. 4 we plot the differences $\epsilon_{max} - \epsilon_0$, $\epsilon_{min} - \epsilon_0$ and $\epsilon_{ave} - \epsilon_0$ for different number m of edges added. For $m > 10$ the differences are all larger than 40, and thus omitted from the figure. The fact that $\epsilon_{min} - \epsilon_0 > 0$ means that average consensus is achieved for all simulated digraphs if ϵ_0 is used. In other words, the upper bounds on parameter ϵ for the randomly generated digraphs (strongly connected and possibly unbalanced) are larger than the bound for cyclic digraphs, and indeed, the more edges are added the larger the bound on ϵ becomes.

We have performed the same simulation as above for initially cyclic digraphs of 9 and 10 nodes (with directed edges randomly added in the same manner as described above), and similar results are observed. This suggests that cyclic digraphs might be the 'worst-case' for convergence of the surplus-based algorithm, in the sense that as long as the parameter satisfies the (tight) bound for cyclic digraphs, this parameter can work for other possibly unbalanced digraphs.

V. CONCLUSIONS

We have studied the continuous-time surplus-based algorithm, and derived a tight upper bound on the parameter included in this algorithm over strongly connected and balanced digraphs. The result guarantees that the states of all agents in the network converge to the average of their initial states when the parameter is smaller than an explicitly computable upper bound.

In future work, we are interested in deriving upper bounds on parameter ϵ for general strongly connected digraphs (possibly unbalanced); in particular we aim to prove the observation, made using simulations, in Section IV-B. We also aim to characterize the relation between parameter ϵ and the convergence speed of algorithm (2).

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