23rd International Symposium on Mathematical Theory of Networks and Systems
Hong Kong University of Science and Technology, Hong Kong, July 16-20, 2018

An action principle for constructing fundamental solution groups for wave equations*

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Abstract— **By exploiting connections between stationary action and optimal control, fundamental solution groups corresponding to a class of wave equations are constructed via dynamic programming.**

Index Terms— **Optimal control, stationary action, dynamic programming, wave equations, fundamental solution groups.**

MSC2010: 35L05, 49J20, 49L20.

I. INTRODUCTION

The *action principle* [1], [2], [3], [4] is a variational principle underpinning modern physics that may be applied to a predefined notion of *action* to yield the equations of motion of a physical system and its underlying conservation laws. An important corollary of the action principle is that *any trajectory of an energy conserving system renders the corresponding action functional stationary in the calculus of variation sense.* With regard to a wave equation that is lossless, the action principle can thus be interpreted as providing a characterization of all of its solutions. This interpretation motivates the development summarized in this extended abstract, in which the action principle is applied via an optimal control representation to construct the corresponding wave equation fundamental solution group.

In order to apply the action principle, compatible notions of kinetic and potential energy are defined with respect to generalized notions of momentum (or velocity) and position that correspond respectively to the input and mild solution of an abstract Cauchy problem [5], [6]. This allows the integrated action to be rigorously defined as a time horizon parameterized functional of the momentum (velocity) input. Unlike the finite dimensional case, this action functional is neither convex nor concave for any time horizon, thereby preventing an immediate generalization of the optimal control approach of [4] to its analysis. As a remedy, a corresponding approximate class of wave equations is considered as an interim step, in which the unbounded linear operator involved is replaced by its (bounded) Yosida approximation. This yields a corresponding action functional that is strictly concave for sufficiently short (but positive) time horizons. The integrated action is subsequently analysed using tools from optimal control theory, semigroup theory, and idempotent analysis, see for example [5], [6], [7], [8], [9]. In particular, an idempotent fundamental solution semigroup applicable on sufficiently short horizons is used to represent the value function of the attendant optimal control problem as an idempotent convolution of a bivariate kernel with a terminal cost. As the characteristics associated with this optimal control problem must correspond to solutions of the approximate wave equation by stationary action, the idempotent fundamental solution semigroup is subsequently used to construct a short horizon prototype for the fundamental solution group for the aforementioned approximate wave equation. These short horizon prototypes are pieced together into long horizon prototypes using the *stat* operation of [10], with the latter converging uniformly to the fundamental solution group of the exact wave equation, in the vanishing limit of the Yosida approximation.

In terms of organization, the class of wave equations of interest is introduced in Section II, with the corresponding action principle formalized via an optimal control problem in Section III. Group construction proceeds in Section IV, and is followed by a brief concluding remark in Section V.

Throughout, $\mathbb{N}$ and $\mathbb{Z}$ denote the natural numbers and integers respectively, while $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ denote the real numbers, non-negative reals, and extended reals. $\mathbb{R}^n$ denotes Euclidean space of dimension $n \in \mathbb{N}$.

II. CLASS OF WAVE EQUATIONS

The wave equations of interest take the abstract form

$$\ddot{x} = -\Lambda\, x \tag{1}$$

in which Λ is an arbitrary linear, unbounded, positive, self-adjoint operator densely defined in an $\mathscr{L}_2$-space $\mathscr{X}$, and possessing a compact inverse $\Lambda^{-1} \in \mathcal{L}(\mathscr{X})$. Spatial boundary conditions for (1) are specified via the domain of Λ, which is denoted by $\mathscr{X}_2 \doteq \mathrm{dom}(\Lambda)$, with $\overline{\mathscr{X}_2} = \mathscr{X}$. It may be noted that this class of wave equations includes those evolving in compact subsets of Euclidean space. For example, selecting Λ to be the additive inverse of the Laplacian operator defined on a Sobolev space of functions compactly supported in $\mathbb{R}^2$ yields such a wave equation [11].

In general, as Λ has a unique, positive, self-adjoint square root $\Lambda^{\frac{1}{2}}$, a pair of useful Hilbert spaces is defined by

$$\mathscr{X}_1 \doteq \mathrm{dom}(\Lambda^{\frac{1}{2}}), \quad \langle x, \xi \rangle_1 \doteq \langle \Lambda^{\frac{1}{2}} x, \Lambda^{\frac{1}{2}} \xi \rangle,$$
$$\mathscr{Y}_1 \doteq \mathscr{X}_1 \times \mathscr{X}, \quad \langle (x,p), (\xi,\pi) \rangle_{\mathscr{Y}} \doteq \langle x, \xi \rangle_1 + \langle p, \pi \rangle,$$

for all $x, \xi \in \mathscr{X}_1$, $p, \pi \in \mathscr{X}$. A solution of (1) is in general interpreted as a mild solution of a corresponding abstract Cauchy problem [5], [6], defined with respect to $\mathscr{Y}_1$ by

$$\begin{pmatrix} \dot{x}_s \\ \dot{p}_s \end{pmatrix} = \mathcal{A} \begin{pmatrix} x_s \\ p_s \end{pmatrix}, \ s \in \mathbb{R}, \quad \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} \in \mathscr{Y}_1. \tag{2}$$

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Here, $\mathcal{A}$ is the unbounded and densely defined operator

$$\mathcal{A} \doteq \left(\begin{array}{c|c} 0 & \mathcal{I} \\ \hline -\Lambda & 0 \end{array} \right), \ \mathrm{dom}(\mathcal{A}) \doteq \mathscr{Y}_2 \doteq \mathscr{X}_2 \times \mathscr{X}_1, \quad (3)$$

that is boundedly invertible, and generates a strongly continuous group $\{\mathcal{U}_t\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathscr{Y}_1)$, see [5, Theorem 4.3, p.14, p.22]. Consequently, a unique continuously differentiable solution of (2) exists on $\mathbb{R}$ for every $(x_0, p_0) \in \mathscr{Y}_2$, and this defines a classical solution of (1), see [5, Theorem 1.3, p.102]. This (and every) solution is given uniquely by

$$\left(\frac{x_s}{p_s} \right) = \mathcal{U}_s \left(\frac{x_0}{p_0} \right), \quad s \in \mathbb{R}. \quad (4)$$

As indicated, the objective is to construct elements of the fundamental solution group $\{\mathcal{U}_t\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathscr{Y}_1)$ for (1). This construction centres on an action principle and its formulation as an optimal control problem.

III. Action principle via optimal control

Define notions of potential and kinetic energy $V : \mathscr{X}_1 \to \mathbb{R}$ and $T : \mathscr{X} \to \mathbb{R}$ respectively by

$$V(x) \doteq \tfrac{1}{2} \|x\|_1^2, \quad T(w) \doteq \tfrac{1}{2} \|\mathcal{J} w\|_1^2 = \tfrac{1}{2} \|w\|^2, \quad (5)$$

for all $x, w \in \mathscr{X}_1$, in which $\mathcal{J} \doteq \Lambda^{-\frac{1}{2}} \in \mathcal{L}(\mathscr{X}_1)$. Given a desired time horizon $t \in \mathbb{R}_{\geq 0}$, define the action functional $J_t : \mathscr{X}_1 \times \mathscr{W}[0,t] \to \mathbb{R}$ with respect to V and T of (5) and an artificial "terminal action" $\psi_v : \mathscr{X}_1 \to \mathbb{R}$ by

$$J_t(x, w) \doteq \int_0^t V(\xi_s) - T(w_s)\, ds + \psi_v(\xi_t), \quad (6)$$

$$\xi_s \doteq [\chi(x, u)]_s = x + \int_0^s w_\sigma\, d\sigma, \quad (7)$$

for all $x \in \mathscr{X}_1$, $w \in \mathscr{W}[0,t]$, $s \in [0,t]$, in which $\mathscr{W}[0,t] \doteq \mathscr{L}_2([0,t]; \mathscr{X}_1)$ and $\psi_v(x) \doteq \langle x, v\rangle_1$ for all $x \in \mathscr{X}_1$, with $v \in \mathscr{X}_1$ fixed. By inspection, initial and terminal data for the dynamics (7) are encoded via x and ψ_v respectively. It may be observed that $J_t(x, \cdot)$, $x \in \mathscr{X}_1$, is neither convex nor concave for any $t \in \mathbb{R}_{>0}$, due to the different norms appearing in (5). However, given arbitrary $\mu \in \mathbb{R}_{>0}$, a concave approximation $J_t^\mu(x, \cdot)$ exists for sufficiently short terminal times $t \in [0, \bar{t}^\mu]$, with $\bar{t}^\mu \doteq \mu\sqrt{2}$, see [12]. This approximation is obtained by replacing Λ in the kinetic energy (5) with its Yosida approximation $\Lambda\, \mathcal{I}_\mu \in \mathcal{L}(\mathscr{X})$ via the Hille-Yosida Theorem [5], where $\mathcal{I}_\mu$ is defined via the resolvent of $-\Lambda$ by $\mathcal{I}_\mu \doteq (\mathcal{I} + \mu^2 \Lambda)^{-1}$. This yields

$$J_t^\mu(x, w) \doteq \int_0^t V(\xi_s) - T^\mu(w_s)\, ds + \psi_v(\xi_t), \quad (8)$$

with $T^\mu : \mathscr{X}_1 \to \mathbb{R}$ defined by $T^\mu(w) \doteq \frac{1}{2} \|(\Lambda\, \mathcal{I}_\mu)^{-\frac{1}{2}} w\|_1^2 = \frac{1}{2}\|w\|^2 + \frac{\mu^2}{2}\|w\|_1^2$ for all $w \in \mathscr{X}_1$, and $(\Lambda\, \mathcal{I}_\mu)^{-\frac{1}{2}}$ exists and is bounded by definition of $\mathcal{I}_\mu$ and boundedness of Λ^{-1}.

Theorem 3.1: [12] Given $\mu \in \mathbb{R}_{>0}$, the value function $W_t^\mu : \mathscr{X}_1 \to \mathbb{R}$ corresponding to (8) is well-defined by

$$W_t^\mu(x) \doteq \sup_{w \in \mathscr{W}[0,t]} J_t(x, w) \quad (9)$$

for all $t \in [0, \bar{t}^\mu]$, $x \in \mathscr{X}_1$.

IV. Group construction

As suggested by Pontryagin's maximum principle, it may be shown [12], [9] that the value function (9) naturally defines (optimal) trajectories and inputs satisfying (7) that render the action functional stationary (in this case, maximal). The objective here is to use these trajectories and inputs, encoded via an idempotent representation of (9), to construct corresponding solutions of the wave equation (1), in the limit as $\mu \to 0$.

A. Short horizons

The idempotent representation of (9) of interest here takes the form of an idempotent convolution of a bivariate quadratic kernel, corresponding to an element of an idempotent fundamental solution semigroup, with the terminal action ψ_v, see [13], [9], [14]. In particular,

$$W_t^\mu(x) = \sup_{y \in \mathscr{X}_1} \{G_t^\mu(x, y) + \psi_v(y)\} \quad (10)$$

for all $x \in \mathscr{X}_1$, $\mu \in \mathbb{R}_{>0}$, $t \in (0, \bar{t}^\mu]$. Here, the kernel $G_t^\mu : \mathscr{X}_1 \times \mathscr{X}_1 \to \overline{\mathbb{R}}$ is defined by $G_t^\mu(x, y) \doteq \frac{1}{2}\langle x, \mathcal{P}_t^\mu x\rangle_1 + \langle x, \mathcal{Q}_t^\mu y\rangle_1 + \frac{1}{2}\langle y, \mathcal{P}_t^\mu y\rangle_1$, in which $\mathcal{P}_t^\mu, \mathcal{Q}_t^\mu \in \mathcal{L}(\mathscr{X}_1)$ denote well-defined self-adjoint and invertible solutions of a pair of operator differential Riccati equations, see for example [9]. As $G_t^\mu(x, \cdot)$ and ψ_v are both Fréchet differentiable, the supremum in (10) is achieved at $y = y_x^* \in \mathscr{X}_1$ satisfying $0 = \mathcal{Q}_t^\mu\, x + \mathcal{P}_t^\mu\, y + v$, or $y_x^* = -(\mathcal{P}_t^\mu)^{-1}\, (\mathcal{Q}_t^\mu\, x + v)$, and the value function (9), (10) satisfies $W_t^\mu(x) = G_t^\mu(x, y_x^*) + \psi_v(y_x^*)$. Meanwhile, the associated optimal trajectory satisfies $\xi_s^* = [\chi(x, w^*)]_s$, $w_s^* = k_t^\mu(s, \xi_s^*)$, for all $s \in [0,t]$, in which $k_t^\mu(s, y) \doteq \Lambda\, \mathcal{I}_\mu \nabla W_{t-s}^\mu(y)$, $s \in [0,t]$, $y \in \mathscr{X}_1$. By inspection, the initial and terminal states and inputs (respectively, generalized positions and velocities or momenta) thus satisfy

$$\begin{aligned} &\xi_0^* = x, && \xi_t^* = y_x^*, \\ &w_0^* = \Lambda\, \mathcal{I}_\mu \nabla W_t^\mu(x), && w_t^* = \Lambda\, \mathcal{I}_\mu \nabla \psi_v(y_x^*) = \Lambda\, \mathcal{I}_\mu v. \end{aligned} \quad (11)$$

Note further, by the chain rule, that

$$\begin{aligned} \nabla_x W_t^\mu(x) &= \nabla_x G_t^\mu(x, y)|_{y=y_x^*} + D_x y_x^*\, \nabla_y G_t^\mu(x, y)|_{y=y_x^*} \\ &= (\mathcal{P}_t^\mu - \mathcal{Q}_t^\mu\, (\mathcal{P}_t^\mu)^{-1}\, \mathcal{Q}_t^\mu)\, x - (\mathcal{P}_t^\mu)^{-1}\, \mathcal{Q}_t^\mu\, v. \end{aligned} \quad (12)$$

Applying (12) in (11), followed by a change of variable $w_s^* \doteq \mathcal{I}_\mu^{\frac{1}{2}}\, \pi_s^*$, $s \in [0,t]$, and some algebraic manipulations, yields

$$\left(\frac{\xi_t^*}{\pi_t^*} \right) = \widehat{\mathcal{U}}_t^\mu \left(\frac{\xi_0^*}{\pi_0^*} \right) = \left(\begin{array}{c|c} [\widehat{\mathcal{U}}_t^\mu]_{11} & [\widehat{\mathcal{U}}_t^\mu]_{12} \\ \hline [\widehat{\mathcal{U}}_t^\mu]_{21} & [\widehat{\mathcal{U}}_t^\mu]_{22} \end{array} \right) \left(\frac{\xi_0^*}{\pi_0^*} \right), \quad (13)$$

for all $t \in (0, \bar{t}^\mu]$, with $\widehat{\mathcal{U}}_t^\mu \in \mathcal{L}(\mathscr{X}_1 \times \mathscr{X})$ defined by

$$\begin{aligned} &[\widehat{\mathcal{U}}_t^\mu]_{11} \doteq -(\mathcal{Q}_t^\mu)^{-1}\, \mathcal{P}_t^\mu, \ [\widehat{\mathcal{U}}_t^\mu]_{12} \doteq (\mathcal{Q}_t^\mu)^{-1}\, \mathcal{E}_\mu^{-1}, \\ &[\widehat{\mathcal{U}}_t^\mu]_{21} \doteq \mathcal{E}_\mu\, \mathcal{Q}_t^\mu\, (\mathcal{I} - [(\mathcal{Q}_t^\mu)^{-1}\, \mathcal{P}_t^\mu]^2), \\ &[\widehat{\mathcal{U}}_t^\mu]_{22} \doteq -\mathcal{E}_\mu\, \mathcal{P}_t^\mu\, (\mathcal{Q}_t^\mu)^{-1}\, \mathcal{E}_\mu^{-1}, \end{aligned} \quad (14)$$

and $\mathcal{E}_\mu \doteq \Lambda\, \mathcal{I}_\mu^{\frac{1}{2}} \in \mathcal{L}(\mathscr{X}_1; \mathscr{X})$, $\mathcal{E}_\mu^{-1} \in \mathcal{L}(\mathscr{X}; \mathscr{X}_1)$. It may be shown [12] that $\{\widehat{\mathcal{U}}_t^\mu\}_{t\in(0,\bar{t}^\mu]}$ is a subset of the uniformly continuous group $\{\mathcal{U}_t^\mu\}_{t\in\mathbb{R}}$ generated by $\mathcal{A}^\mu \in \mathcal{L}(\mathscr{Y}_1)$, where

$$\mathcal{A}^\mu \doteq \left(\begin{array}{c|c} 0 & \mathcal{I}_\mu^{\frac{1}{2}} \\ \hline -\Lambda\,\mathcal{I}_\mu^{\frac{1}{2}} & 0 \end{array}\right), \quad \mathrm{dom}(\mathcal{A}^\mu) \doteq \mathscr{Y}_1. \tag{15}$$

Analogously to (2), (3), (4), observe that $\{\mathcal{U}_t^\mu\}_{t\in\mathbb{R}}$ defines solutions of the corresponding abstract Cauchy problem

$$\left(\frac{\dot{\xi}_s}{\dot{\pi}_s}\right) = \mathcal{A}^\mu \left(\frac{\xi_s}{\pi_s}\right), \ s \in \mathbb{R}, \quad \left(\frac{\xi_0}{\pi_0}\right) \in \mathscr{Y}_1, \tag{16}$$

and that these are always classical solutions. Consequently, by inspection of (15), (16), the approximate action principle encapsulated via (8), (9) yields solutions of

$$\ddot{\xi} = -\Lambda\,\mathcal{I}_\mu\, x, \tag{17}$$

which is also a wave equation. As $-\Lambda\,\mathcal{I}_\mu \in \mathcal{L}(\mathscr{X}_1)$ is the Yosida approximation of $-\Lambda$, note that $-\Lambda\,\mathcal{I}_\mu$ converges strongly to $-\Lambda$ as $\mu \to 0^+$ on $\mathrm{dom}(-\Lambda) = \mathscr{X}_2$, see [5, Lemma 3.3, p.10]. In this formal sense, the wave equation (17) can be considered as converging to (1) in the limit. Consequently, it is referred to as the approximate wave equation corresponding to (1). A statement concerning convergence of solutions of (17) to those of (1) requires an application of the Trotter-Kato theorem, see [5, Theorem 4.4, p.87], [9].

B. *Longer horizons*

The prototype fundamental solution group $\{\widehat{\mathcal{U}}_t^\mu\}_{t\in(0,\bar{t}^\mu]}$ defined by (14), and constructed via the action principle encapsulated by (8), (9), is a subset only of the corresponding fundamental solution group $\{\mathcal{U}_t^\mu\}_{t\in\mathbb{R}}$ for the approximate wave equation (17). Furthermore, the time horizon $\bar{t}^\mu = \mu\sqrt{2}$ on which it is defined converges to zero as $\mu \to 0^+$. Consequently, in order to construct the fundamental solution group for (17), and hence (1), an extension to longer horizons is required. As the short horizon restriction evident is due to a loss of concavity of the action functional (8) on longer horizons, it is crucial that concavity be relaxed to stationarity at some intermediate times. This can be achieved via suitable use of the *stat* operation [10] in defining an analog of the value function (9) on longer horizons, via a generalization of the convolution representation (10). Indeed, motivated by the associated quadratic form for the kernel G_t^μ, it is possible to define a long horizon value function [12] by

$$\widetilde{W}_t^\mu(x) \doteq \underset{y\in\mathscr{X}_1}{\mathrm{stat}} \left\{ \widetilde{G}_t^\mu(x,y) + \psi_v(y) \right\} \tag{18}$$

for all $t \in \Omega^\mu$, $x \in \mathscr{X}_1$, in which Ω^μ is an unbounded, uncountable, and dense subset of $\mathbb{R}_{\geq 0}$ selected so as to avoid finite escape times of the associated Riccati equations for all $\mu \in \mathbb{R}_{\geq 0}$, see [12], and $\widetilde{G}^\mu$ is defined via an iterative concatenation of short horizons. In particular, given $n_t \in \mathbb{N}$ sufficiently large such that $\tau \doteq t/n_t \in (0, \bar{t}^\mu]$,

$$\widetilde{G}_{k\tau}(x,y) \doteq \underset{\eta\in\mathscr{X}_1}{\mathrm{stat}} \left\{ \widetilde{G}_{(k-1)\tau}^\mu(x,\eta) + G_\tau^\mu(\eta,y) \right\}, \tag{19}$$

for all $k \in \mathbb{N}_{\leq n_t}$, in which G_τ^μ is the quadratic form as per (10). By choice of Ω^μ, (19) is also a quadratic form, and indeed has exactly the same analytical expression as the short horizon case. Following analogous steps as per (10), (11) yields that the subset $\{\widehat{\mathcal{U}}_t^\mu\}_{t\in(0,\bar{t}^\mu]}$ of the approximate fundamental solution group defined by (14) may be extended to $\{\widetilde{\mathcal{U}}_t^\mu\}_{t\in\Omega^\mu}$, while satisfying $\widetilde{\mathcal{U}}_t^\mu = \widehat{\mathcal{U}}_t^\mu$ for all $t \in (0, \bar{t}^\mu]$. This extended subset yields all solutions of the approximate wave equation (17), with those solutions being defined almost everywhere via Ω^μ, and taking the same form as (13). An application of the Trotter-Kato theorem [5, Theorem 4.4, p.87] subsequently yields that these solutions converge uniformly on bounded intervals to those of the wave equation (1) of interest.

Theorem 4.1: There exists an unbounded, uncountable, and dense $\Omega \subset \mathbb{R}$, and a sequence $\{\mu_j\}_{j\in\mathbb{N}} \in \mathbb{R}_{>0}$ satisfying $\lim_{j\to\infty} \mu_j = 0$, such that

$$0 = \lim_{j\to\infty} \left\| \widetilde{\mathcal{U}}_t^{\mu_j} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} - \mathcal{U}_t^{\mu_j} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} \right\|_{\mathscr{Y}}$$

for all $(x_0, p_0) \in \mathscr{Y}_1$, uniformly in t on bounded subsets of Ω.

V. Conclusions

A fundamental solution group is constructed for a general class of lossless wave equations via an action principle and its encapsulation within an optimal control problem.

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