

Opinion Dynamics with State-Dependent Susceptibility to Influence

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Abstract—A general discrete-time opinion dynamics model is studied in this paper. It is proposed that each individual has a susceptibility to being influenced by his/her neighbours, and that this susceptibility depends on the individual's current opinion value. This precept is captured by a state-dependent susceptibility function. Two different susceptibility functions are proposed to describe individuals who are stubborn conformists and stubborn extremists. Stubborn conformists are individuals who become more closed to influence when they have an opinion similar to the network average, while stubborn extremists are less susceptible when they hold opinions at either end of the opinion interval. Convergence results are established for networks where all individuals are stubborn conformists, or all individuals are stubborn extremists. Simulations are provided to illustrate the results. Key conclusions, consistent with sociology literature and exemplified by the simulations are that (i) stubborn conformists typify observed phenomenon whereby it takes a long time for people to agree on social norms, and an equally difficult time breaking them, and (ii) existence of stubborn extremists can pull individuals with an initial neutral opinion to the extremes.

I. INTRODUCTION

Among the many different areas of research on social networks, the topic of opinion dynamics has been one of the most popular. A key focus has been to develop agent-based mathematical models to describe this evolution; the models are required to be well grounded with established and accepted sociological and social psychological processes. The discrete-time French-Harary-DeGroot model [1], [2] (known commonly as the DeGroot model) is perhaps the most widely accepted and studied model, and proposes that each individual updates his/her opinion to be equal to a convex combination of his/her own opinion, and the opinion of his/her neighbours. That is, the neighbours are able to exert an influence on the individual. Under mild assumptions on the network structure, the opinions reach a consensus. Since then, many models have been proposed to capture various observed social phenomenon, both in continuous- and discrete-time setting. See [3]–[5] for a survey of works.

In the DeGroot model and its variants, a key aspect is the interpersonal influence, i.e. the amount of influence an

individual's neighbour is able to exert on the individual and thereby affect the individual's new opinion value. Subsequent works often assume that influence can depend on the individual's opinion value and/or the opinion value of the individual's neighbour. The Friedkin-Johnsen model [3], [6] assumes that each individual may remain partially attached to his/her initial opinion, and is therefore partially closed to being influenced by his/her neighbours. The Hegselmann-Krause model [7]–[9] popularised the bounded-confidence approach to describing the social process of homophily, whereby individuals are influenced only by those others who have similar opinion values. In [10], bias assimilation affects influence; an individual puts more weight on opinions closely aligned with his/her own opinion. The Altafini model [11], [12] incorporates negative weights to describe antagonistic influence between hostile individuals.

Recently, a new continuous-time model was proposed in [13], where the influence of the individual's neighbours depended on how polarised the individual's current opinion value was. A susceptibility function is used in the model to capture how susceptible the individual is to influence from his/her neighbours. Different susceptibility functions are proposed. Stubborn neutrals are individuals who become increasingly closed to influence as their opinions approach a neutral stance. Stubborn positives are individuals who become increasingly closed to influence as their opinions approach one extreme of the opinion interval, but not the other. Lastly, stubborn extremists are individuals who become increasingly closed to influence when at either extreme of the opinion interval. A discrete-time counterpart was developed by the authors [14] and limiting behaviour was established for networks with individuals who were either stubborn neutral, or stubborn positive.

The current work, though inspired by ideas of [13], [14], differs in key respects. First, we propose a new type of susceptibility function to describe stubborn conformist individuals, i.e. those individuals who become more closed to influence as they reach the average group opinion. Examples from sociology literature are provided for motivation. We show that the social network globally asymptotically reaches a consensus, but not exponentially so. We also characterise the limiting behaviour of the discrete-time model with stubborn extremist individuals. Specifically, we show that whenever there are no individuals with initial opinions at either extremes of the opinion interval, a consensus of opinions is reached exponentially fast. When there is at least one individual with initial opinion at one extreme and no individuals with initial opinions at the other extreme, the network asymptotically reaches a consensus at that extreme

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opinion. If there are individuals with initial opinions at both extremes of the opinion interval, we establish the existence of a unique equilibrium (without establishing convergence). Simulations are given to show the limiting behaviour.

Immediately below, we present preliminaries and notations. The rest of the contribution is structured as follows. The opinion dynamics model is introduced in Section II, and the main results on stubborn conformists and stubborn extremists are given in Section III. Simulations are provided in Section IV and concluding remarks appear in Section V.

A. Preliminaries

Given a positive integer n , we use $[n]$ to denote the index set $\{1, 2, \dots, n\}$. Vectors are viewed as column vectors and x^\top denotes the transpose of a vector x . For a vector x , we use x_i to denote the i th entry of x . For any matrix M , m_{ij} denotes its ij th entry. A square matrix with nonnegative entries is called a stochastic matrix if its row sums are all equal to 1. We use $\mathbf{0}$ and $\mathbf{1}$ to denote the vectors whose entries all equal to 0 and 1, respectively, and I to denote the identity matrix, while the dimensions of the vectors and matrices are to be understood from the context. For $x \in \mathbb{R}$, $|x|$ is used to denote the absolute value of x . The graph of an $n \times n$ real matrix M is an n -vertex directed graph defined so that (i, j) is an arc from vertex i to vertex j in the graph whenever the ij th entry of M is nonzero. We will use the terms “individual” and “agent” interchangeably.

II. BACKGROUND ON THE MODEL

In a recent paper, a continuous-time generalised model of opinion dynamics with state-dependent susceptibility was proposed [13]. The model was termed “polar opinion dynamics” to reflect the fact that each individual’s susceptibility to influence was captured by a “susceptibility function” that depends on how polarised his/her opinion value is. A discrete-time counterpart was proposed in [14], and some limiting behaviour was established. Here, we briefly reintroduce the model before presenting new results.

Consider a social network of $n > 1$ individuals, labelled 1 through n , discussing opinions on a given topic.¹ Each agent i can only learn, and be influenced by, the opinions of certain other agents called the neighbours of agent i . Neighbour relationships among the n agents are described by a directed graph \mathbb{N} , called the neighbour graph. Agent j is a neighbour of agent i if (j, i) is an arc in \mathbb{N} . Thus, the directions of arcs indicate the directions of information flow (specifically opinion flow) and interpersonal influence. For convenience, we assume that each individual is a neighbour of himself/herself, i.e. \mathbb{N} has self-arcs at all n vertices. The set neighbours of agent i is denoted by \mathcal{N}_i . Each individual i has an opinion, a real-valued quantity, x_i , on a given topic.

¹The labelling of the individuals is for convenience purposes only. A global labelling of the individuals in the network is not required. We only assume that each agent can identify his/her own neighbours.

Each individual updates his/her opinion simultaneously according to the following rule:

$$x_i(t+1) = x_i(t) + f_i(x_i(t)) \sum_{j \in \mathcal{N}_i} w_{ij}(x_j(t) - x_i(t)) \quad (1)$$

where the function $f_i(x_i(t)) \in [0, 1]$ captures individual i ’s susceptibility to having his/her opinion $x_i(t)$ changed via the influence of others’ opinions. Apart from the susceptibility function, the influence weight w_{ij} indicates how much individual i is influenced by individual j , and we require that $\sum_{j=1}^n w_{ij} = 1$ for all $i \in [n]$. For convenience, we denote $u_i(t) = \sum_{j \in \mathcal{N}_i} w_{ij}(x_j(t) - x_i(t))$ as the influence from i ’s neighbours, whose effect on the determination of $x_i(t+1)$ is adjusted by i ’s susceptibility $f_i(x_i(t))$. Clearly, if $f_i(x_i(t)) = 1$ then for time instant t , individual i updates his/her opinion according to the DeGroot model, whereas if $f_i(x_i(t)) = 0$, then for time instant t , individual i is totally closed to external influence and $x_i(t+1) = x_i(t)$. Additional details of the model derivation, its relation to (i) the continuous-time model studied in [13], (ii) the seminal DeGroot model [2] and (iii) the Friedkin-Johnsen model [6] are discussed in [14, Section II].

One can write the compact form of the opinion dynamical system for n individuals in the social network as

$$\begin{aligned} x(t+1) &= (I - F(x(t)))x(t) + F(x(t))Wx(t) \quad (2) \\ &= S(x(t))x(t) \quad (3) \end{aligned}$$

where $F(x(t))$ is a diagonal matrix with i th entry $f_i(x_i(t))$, W is the influence matrix with the ij th entry being the influence weight w_{ij} , and $x(t) = [x_1(t), \dots, x_n(t)]^\top$ being the vector of individuals’ opinions. It was verified in [14] that $S(x(t))$ is a row-stochastic matrix for all $t \geq 0$.

In this contribution, and as in [13], [14], it is assumed that the initial opinions satisfy $x_i(0) \in [-1, 1]$, for all $i \in [n]$. Here, -1 and 1 represent the extreme negative and positive opinions, respectively. Such a scaling is typical in opinion dynamics problems, e.g. x_i may represent individual i ’s attitude towards an idea. An example is to suppose the topic of discussion is whether “the new iPhone is worth buying” and thus $x_i = 1$ is maximally supporting, $x_i = 0$ is neutral, and $x_i = -1$ maximally opposing of the statement. The following results were established regarding the boundedness of the opinion vector $x(t)$.

Lemma 1 ([14, Lemma 1 and 2]). *Suppose that each agent i follows the update rule (1) and that $x_i(0) \in [-1, 1]$ for all $i \in [n]$. Then, $x_i(t) \in [-1, 1]$ for all $i \in [n]$ and time t , and $x_{\min}(t) = \min_{i \in [n]} x_i(t)$ is nondecreasing and $x_{\max}(t) = \max_{j \in [n]} x_j(t)$ is nonincreasing as t increases.*

These results establish that (i) $[-1, 1]$ is an invariant set of each individual’s opinion dynamics model (1), and (ii) the most negative and positive opinions in the social network will never become more negative and more positive, respectively.

We have introduced the general opinion dynamics model, in which the state-dependent susceptibility function $f_i(x_i(t))$ affects the amount of influence individual i ’s neighbours have

on determining $x_i(t + 1)$. In the next section, we introduce specific function forms for $f_i(x_i(t))$ to describe stubborn conformist and stubborn extremist individuals.

III. MAIN RESULTS

Two specific susceptibility functions are now considered and the behaviour of the corresponding opinion dynamics models are given. Motivating examples from sociology are provided to justify the functional forms. These two functions were not studied by the authors in [14], though one was studied in the continuous-time model in [13]. Proofs are omitted due to space limitations.

A. Stubborn Conformists

An individual i is said to be a stubborn conformist if

$$f_i(x_i(t), x(t)) = \frac{|x_i(t) - \bar{x}(t)|}{2} \quad (4)$$

where $\bar{x}(t) = \mathbf{1}^\top x(t)/n$ is the global average opinion value at time instant t . In this subsection, we assume that every individual $i \in [n]$ is a stubborn conformist. Note that this type of individual was not considered in [13], [14].

Motivation: Notice that for stubborn conformists, individual i 's susceptibility to influence decreases as $x_i(t)$ approaches $\bar{x}(t)$, the average opinion of the social network, and increases as $x_i(t)$ moves away from $\bar{x}(t)$. Such a susceptibility function reflects a social pressure to conform to the group (social network) norm or average. In Solomon E. Asch's seminal work [15], he showed that an individual could feel overwhelming pressure to follow the group majority opinion, i.e. \bar{x} , even if the individual knew the opinion was flawed. High productivity factory workers are sometimes pressured to lower production rates to conform with the averages of their co-workers [16]. In [17], it was shown that individuals in a shared housing tended to express opinions that were *modified due to normative pressure* to be closer to the average of the other tenants. The recent paper [18] proposed an opinion dynamics model where individuals had a preference for conforming with the average opinion. However, the model is extremely complex, and is analysed via simulations whereas we are able to provide a complete convergence result. To summarise, stubborn conformists are individuals who become closed to influence when they observe that their opinions are similar to the majority of the social network, i.e. $\bar{x}(t)$. Such situations can often arise during the discussion of politically sensitive topics such as gender equality, or same-sex marriage.

We now establish the limiting behaviour of the system (2) with individuals having susceptibility functions given by (4).

Theorem 1. *Suppose that the neighbour graph \mathbb{N} is strongly connected, and $x_i(0) \in [-1, 1], \forall i \in [n]$. Then, the system (2), with each individual $i \in [n]$ having a susceptibility function given by (4), asymptotically reaches a consensus, i.e. $\lim_{t \rightarrow \infty} x(t) = \alpha \mathbf{1}, \alpha \in [-1, 1]$.*

Remark 1. *It is possible to show that the convergence to consensus is asymptotic, but not exponential, and is related*

to the fact that $\lim_{x_i \rightarrow \bar{x}} f_i(x_i(t), x(t)) = 0$. This can also be seen in the simulations in Section IV. One way to interpret this observation is that getting everyone to agree precisely on the same opinion is a difficult and slow process, but equally, attempts to change people's opinions once they are roughly near $\bar{x}(t)$ are correspondingly more difficult.

B. Stubborn Extremists

Consider now individual i having susceptibility function

$$f_i(x_i(t)) = 1 - x_i(t)^2 \quad (5)$$

We call such an individual a stubborn extremist, and note that such a function was first introduced and motivated in [13]. The motivations provided below are therefore shorter when compared with the stubborn conformists (which is new).

Motivation: Stubborn extremists become less susceptible to influence as $x_i(t)$ approaches the two extreme ends of the opinion interval $[-1, 1]$. They are fully open to influence when $x_i(t) = 0$, taking a neutral opinion value. That is, stubborn extremists are individuals whose strength of conviction increases as they become more extreme in their position. Sociology and social psychology literature points to the fact that individuals become more resistant to change of opinion as they become more polarised, or extreme, in their stance [19]–[21]. One can see that this is particularly likely to arise during the discussion of topics for which there are two competing positions, e.g. Democrats versus Republicans or iPhone versus Android.

The following theorem characterizes some limiting behaviour of the system (2) with stubborn extremist individuals.

Theorem 2. *Suppose that the neighbour graph is strongly connected, and that every individual $i \in [n]$ has $x_i(0) \in [-1, 1]$ and a susceptibility function given by (5). If $|x_i(0)| \neq 1$ for all $i \in [n]$, then all $x_i(t)$ in (1) will reach a consensus exponentially fast at some value in the interval $(-1, 1)$. If $x_i(0) \neq -1$ for all $i \in [n]$ and there exists at least one $j \in [n]$ such that $x_j(0) = 1$, then all $x_i(t)$ in (1) will asymptotically reach a consensus at 1. If $x_i(0) \neq 1$ for all $i \in [n]$ and there exists at least one $j \in [n]$ such that $x_j(0) = -1$, then all $x_i(t)$ in (1) will asymptotically reach a consensus at -1 .*

Remark 2. *Theorem 2 does not cover initial conditions where there exists $j, k \in [n]$, with $j \neq k$, such that $x_j(0) = 1$ and $x_k(0) = -1$. Under these circumstances, we conjecture that similar to the continuous-time result [13], the social network will reach an equilibrium where there is disagreement among the opinions. This is because under such initial conditions, we have two totally stubborn individuals at either end of the opinion interval. While we can show that the system (2) with stubborn extremists has a unique equilibrium point x^* inside the hypercube $(-1, 1)^n$, we have not yet been able to establish convergence to this equilibrium, though all simulations showed convergence.*

IV. SIMULATIONS

We now provide a simulation example to illustrate the effects of the susceptibility function $f_i(x_i(t))$. A strongly

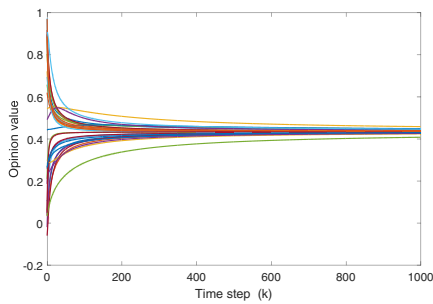


Fig. 1. Opinions of stubborn conformists $f_i(x_i(t)) = \frac{|x_i(t) - \bar{x}(t)|}{2}$.

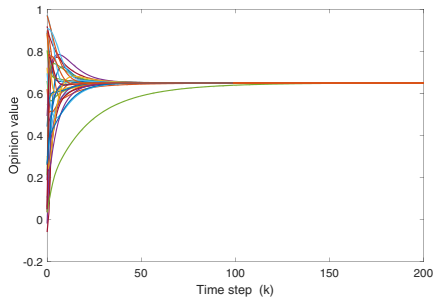


Fig. 2. Opinion evolution of stubborn extremists $f_i(x_i(t)) = 1 - x_i(t)^2$.

connected social network with $n = 30$ individuals is generated with randomly selected influence weights w_{ij} . The W matrix is not shown due to spatial limitations. The initial conditions are sampled from a uniform distribution in the interval $(-0.1, 1)$. For the same graph and initial conditions, we simulated 1) the opinion dynamics with stubborn conformists $f_i(x_i(t)) = \frac{|x_i(t) - \bar{x}(t)|}{2}, \forall i \in [n]$, and 2) the opinion dynamics with stubborn extremists $f_i(x_i(t)) = 1 - x_i(t)^2, \forall i \in [n]$. The results for stubborn conformist and stubborn extremist dynamics are shown in Fig. 1 and Fig. 2, respectively. Simulations of the original DeGroot model, stubborn neutrals, and stubborn positives are in [14].

While consensus is reached in both simulations, clearly there is a difference between the asymptotic (but not exponential) convergence of stubborn conformists in Fig. 1 and the exponential convergence of stubborn extremists in Fig. 2. Also observed in Fig. 2 is the effect of individuals with initial $x_i(0)$ close to 1 (meaning they are more closed to influence from neighbours j with $x_j(t)$ close to 0); the final consensus value is much closer to 1, when compared to Fig. 1.

V. CONCLUSIONS

In this contribution, we have revisited and further developed a recently introduced [14] discrete-time opinion dynamics model with state-dependent susceptibility to influence. Susceptibility functions have been introduced used to describe individuals who were stubborn conformists, or stubborn extremists, with sociology literature provided as motivation. Convergence results have then been established, and simulations have been provided to illustrated the results. Future work will involve generalisation to allow for

heterogeneous susceptibility functions belonging to a class of functions, and to consider time-varying networks. In addition, the convergence of the case discussed in Remark 2 is to be established.

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