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Exponential Synchronization of Lohe Oscillators with Strongly Connected Topologies

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Abstract— **Lohe model is a typical dynamic network of nonlinear multi-agent systems. Under strongly connected topologies, the synchronization problem of Lohe oscillators is investigated. The obtained results show that, limited on a half unit sphere, synchronization of Lohe model can be achieved. To analyze exponential synchronization, a matrix Riccati differential equation of synchronization errors is proposed for the first time. It is proved that, after a finite amount of time, the synchronization errors converge to zero exponentially.**

Index Terms— **Lohe model, exponential synchronization, strongly connected topology.**

I. INTRODUCTION

Synchronization behavior is a common phenomenon in biological [1], [2], ecological [3] and mechanical systems [4]. There are many mathematical models to describe the phenomenon such as Boid model [5], Vicsek model [6] and Kuramoto model[7]. The dynamic equations of the Kuramoto model composed of m oscillators are described as follows:

$$\dot{\theta}_i = \omega_i + k \sum_{j=1}^{m} a_{ij} \sin(\theta_j - \theta_i), \quad i = 1,2,\cdots,m, \tag{1}$$

where θ_i's are the phase angles, ω_i's are the natural frequencies, $A = (a_{ij})$ is the nonnegative adjacency matrix of the interconnecting network and $k > 0$ is the control gain. Kuramoto model has been applied to many fields such as neuro-science [8], power systems [9] and chemical engineering [10]. Elegant summaries on synchronization of Kuramoto model can be found in [11] and [12]. It is well-known that Kuramoto model actually describes a collective behavior on the unit circle of the plane. A generalized form of Kuramoto model on the unit sphere of a high-dimensional linear space, called high-dimensional Kuramoto model or Lohe model, is

$$\dot{r}_i = \Omega_i r_i + k \sum_{j=1}^{m} a_{ij} (r_j - \frac{r_i^T r_j}{r_i^T r_i} r_i), \ i = 1,2,\cdots,m, \tag{2}$$

where $r_i \in \mathbf{R}^n$ is the state of oscillator i, W_i is a real $n \times n$ skew-symmetric matrix, $k > 0$ is the control gain and $A = (a_{ij}) \in \mathbf{R}^{m\times m}$ is the adjacency matrix of the interconnecting network. In the pioneering literatures [13], [14], Lohe gave many numerical simulations to verify the collective dynamical behaviors of (2). For the case of $\Omega_i = 0$ in (1), Olfati-Saber first provided rigorous mathematical proof for synchronization of Lohe model under a topology of the complete graph [15]. In our earlier work [16], we investigated the synchronization of Lohe model under general undirected graph. By using LaSalle invariance principle, it is proved that synchronization can be achieved if the topology is a connected undirected graph and all the initial states are limited on a half unit sphere. In [17], a more general high-dimensional Kuramoto model defined on a curved surface is investigated. In [18] and [19], for the topology of complete graph, exponential synchronization is proved by using the concept of order parameter. A natural problem is how to achieve exponential synchronization for Lohe model under general topologies instead of a complete graph.

In this paper, the synchronization of Lohe model under a general undirect network limited on a half sphere is achieved. The synchronization errors are described by a matrix Riccati differential equation, by which a sufficient condition for exponential synchronization is achieved for the case of general directed topolgies.

The rest of this paper is organized as follows. Section 2 includes our main results. Section 3 shows a simulation. Finally, Section 4 is devoted to a summary of our main results.

II. MAIN RESULT

Consider the Lohe model under the topology described by a directed graph $G = (V,E,A)$, which is composed a set of nodes $V = \{1,2,\cdots,m\}$, set of edges $E \subset V \times V$ and a adjacent matrix $A = (a_{ij}) \in \{0,1\}$. An edge (i,j) mean that agent j can receive the state information of agent i. Adjacency matrix defined that

$$a_{ij} = \begin{cases} 1, & (j,i) \in E \\ 0, & \text{otherwise.} \end{cases}$$

The Laplace matrix $L = (l_{ij})$ of a digraph G defined by

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum\limits_{k\neq i} a_{ik}, & i = j. \end{cases}$$

Lemma 1: (Corollary 3 of [21]) Let $G = (V,E)$ be a directed graph with Laplacian matrix L. If G is strongly connected, then L has a simple zero eigenvalue and a positive left-eigenvector associated to the zero eigenvalue.

In this paper, we only consider the Lohe model with identical oscillators.

Lemma 2: (Proposition 1 of [16]) Consider the Lohe model (2) with $\Omega_i^T = -\Omega_i$, the value of $\|r_i\|$ is a constant for any $t \geq 0$ and $i = 1,2,\cdots,m$.

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If $\Omega_i = \Omega$ for each $i = 1, 2, \cdots, m$, then with the transformation $z_i(t) = \mathrm{e}^{-\Omega t} r_i(t)$, we have

$$\dot{z}_i = k \sum_{j=1}^{m} a_{ij}(z_j - (z_i^T z_j) z_i). \tag{3}$$

So, without loss of generality, we consider the Lohe model with each r_i limited on the unit sphere as follows:

$$\dot{r}_i = k \sum_{j=1}^{m} a_{ij}(r_j - (r_i^T r_j) r_i). \tag{4}$$

A. Synchronization of Lohe model under directed topologies

Lemma 3: Let $v \in \mathbf{R}^n$ be a fixed vector and ε be a small positive number. Set

$$S_v^\varepsilon = \{r = (r_1^{\mathrm{T}},\ r_2^{\mathrm{T}}, \cdots, r_m^{\mathrm{T}})^{\mathrm{T}} \in \mathbf{R}^{mn} \mid v^{\mathrm{T}} r_i \geq \varepsilon, \\ \|r_i\| = 1,\ i = 1, 2, \cdots, m.\}. \tag{5}$$

Then S_v^ε is a positively invariant compact set of Lohe model (4).

Proof: Let $h(t) = \min_{1 \le i \le m} v^{\mathrm{T}} r_i(t)$ and $P_t = \{i | h_i(t) = h(t)\}$. For any given $t \geq 0$, we assume $r(t) \in S_v^\varepsilon$. Since $h(t)$ is a non-smooth min-function, we calculate its Dini derivative by Lemma 2.2 in [22] as follows:

$$\begin{aligned}
D^+ h(t) &= \min_{i \in P_t} \dot{h}_i(t) \\
&= \min_{i \in P_t} v^{\mathrm{T}} \left(k \sum_{j=1}^{m} a_{ij}(r_j(t) - (r_i^T(t) r_j(t)) r_i(t)) \right) \\
&= \min_{i \in P_t} k \sum_{j=1}^{m} a_{ij}(v^{\mathrm{T}} r_j(t) - (r_i^T(t) r_j(t)) v^{\mathrm{T}} r_i(t)) \\
&\geq \min_{i \in P_t} k \sum_{j=1}^{m} a_{ij}(v^{\mathrm{T}} r_j(t) - v^{\mathrm{T}} r_i(t)) \\
&= \min_{i \in P_t} k \sum_{j=1}^{m} a_{ij}(v^{\mathrm{T}} r_j(t) - h(t)) \\
&\geq 0.
\end{aligned} \tag{6}$$

Thus, $h(t)$ is a nondecreasing function. So, if $r(0) \in S_v^\varepsilon$, then $v^{\mathrm{T}} r_i(t) > 0$ for all $t \geq 0$ and $i = 1, 2, \cdots, m$. Moreover, by Lemma 2, we have $r_i(t) = 1$ for any $t \geq 0$. Therefore, S_v^ε is a positively invariant set of (4). ■

Theorem 1: Consider the Lohe model (4) with $k > 0$ and the weighted adjacency matrix $A = (a_{ij})$ of the digraph $\mathscr{G}$. Suppose that $\mathscr{G}$ is strongly connected and there exists $v \in \mathbf{R}^n$ such that $v^{\mathrm{T}} r_i(0) > 0$ for every $i = 1, 2, \cdots, m$. Then $r(t)$ converges to a synchronization point, that is, there exists $\bar{r} \in \mathbf{R}^n$ such that $\lim_{t \to +\infty} r_i(t) = \bar{r}$ for every $i = 1, 2, \cdots, m$.

Proof: Let $s_i(t) = v^{\mathrm{T}} r_i(t)$ for every $i = 1, 2, \cdots, m$ and $t \geq 0$. From the limitation on the initial conditions, it follows that there exists $\varepsilon > 0$ such that

$$s_i(0) = v^{\mathrm{T}} r_i(0) \geq \varepsilon \quad \forall\ i = 1, 2, \cdots, m, \tag{7}$$

which implies that each $r_i(0)$ belongs to S_v^ε denoted by (5) in Lemma 3. Thus, by the invariance of S_v^ε in Lemma 3, we have

$$s_i(t) = v^{\mathrm{T}} r_i(t) \geq \varepsilon > 0, \quad \forall\ i = 1, 2, \cdots, m,\ \forall\ t \geq 0. \tag{8}$$

Considering Lohe model (4), we get the dynamical equations of s_i as

$$\begin{aligned}
\dot{s}_i &= k \sum_{j=1}^{m} a_{ij} v^{\mathrm{T}} (r_j - r_j^T r_i r_i) \\
&= k \sum_{j=1}^{m} a_{ij} s_j - k \sum_{j=1}^{m} a_{ij} r_j^{\mathrm{T}} r_i s_i \\
&= k \sum_{j=1}^{m} a_{ij}(s_j - s_i) + k \sum_{j=1}^{m} a_{ij}(1 - r_j^T r_i) s_i,
\end{aligned} \tag{9}$$

which can be rewritten in the compact form

$$\dot{s} = -kLs + kF(r), \tag{10}$$

where $s = (s_1, s_2, \cdots, s_m)^T$, L is the Laplacian matrix of $\mathscr{G}$ and

$$F(r) = \begin{pmatrix} \sum_{j=1}^{m} a_{1j}(1 - r_j^T r_1) s_1 \\ \sum_{j=1}^{m} a_{2j}(1 - r_j^T r_2) s_2 \\ \vdots \\ \sum_{j=1}^{m} a_{mj}(1 - r_j^T r_m) s_m \end{pmatrix}. \tag{11}$$

Since digraph $\mathscr{G}$ is strongly connected, by Lemma 1, we have that the Laplacian matrix L has a positive left eigenvector associated with the zero eigenvalue, i.e. there exists vector $\beta = (\beta_1, \beta_2, \cdots, \beta_m)^T \in \mathbf{R}^m$ s.t. $\beta^T L = 0$ with all $\beta_i > 0$ ($i = 1, 2, \cdots, m$). Let

$$V(r) = -\beta^T (r_1^T v, r_2^T v, \cdots, r_m^T v)^T = -\beta^T s.$$

It is easy to see

$$\begin{aligned}
\dot{V} &= -\beta^T \dot{s} \\
&= k\beta^T L s - k\beta^T F \\
&= 0 - k \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}(1 - r_j^T r_i) \beta_i s_i \\
&\leq 0.
\end{aligned} \tag{12}$$

In the following, we assume that the equality in (12) holds. Then

$$a_{ij}(1 - r_j^T r_i) \beta_i s_i = 0 \quad (i, j = 1, 2, \cdots, m). \tag{13}$$

Since $\beta_i > 0$ and $s_i > 0$, it follows from (13) that $a_{ij} > 0$ implies $r_i = r_j$. In other words, if (j, i) is an edge, then $r_i = r_j$. Therefore, by the strong connectedness of graph $\mathscr{G}$, we have $r_1 = r_2 = \cdots = r_m$. Thus

$$\{r | \dot{V} = 0\} = \{r | r_1 = \cdots = r_m\}. \tag{14}$$

Moreover, we have an invariant compact set as follows:

$$\Omega = \{r = (r_1^{\mathrm{T}}, r_2^{\mathrm{T}}, \cdots, r_m^{\mathrm{T}})^{\mathrm{T}} \in \mathbf{R}^{mn} \mid r_i \in \mathbf{R}^n, \\ \|r_i\| = 1,\ i = 1, 2, \cdots, m\}. \tag{15}$$

Thus, by LaSalle Invariance Principle, $r(t)$ converges to a consensus point. ■

Remark 1: The geometric meaning of $v^{\mathrm{T}} r_i(0) > 0$ is that the angle between vectors v and $r_i(0)$ is less than 90 degrees. The assumption of $v^{\mathrm{T}} r_i(0) > 0$ ($i = 1, 2, \cdots, m$) means that all the initial vectors lie in an open semi-sphere centered at the origin, which is helpful to construct a compact invariant set S_v^{ε} for a sufficiently small number $\varepsilon > 0$.

Remark 2: Compared with the existing results on synchronization of Lohe model [18], [19], we achieve the synchronization result for general directed graphs instead of a complete graph. Our method is completely different from that used in [18] and [19]. Although we do not prove the exponential synchronization in Theorem 1, we have weakened the topology condition to a directed graph. In the next subsection, we try to prove the exponential synchronization.

B. Exponential synchronization of Lohe model

Let $e_{ij} = 1 - r_i^T r_j$. It is easy to see that $e_{ij} = e_{ji}$, $e_{ii} = 0$ and $0 \le e_{ij} \le 2$ for any $i, j = 1, 2, \cdots, n$. Obviously, $e_{ij} = 0$ if and only if $r_i = r_j$. So e_{ij} reflects the error between r_i and r_j. In the following, let us investigate the dynamics of all the e_{ij}'s. With simple calculations, we have

$$\begin{aligned}
\dot{e}_{ij} &= -r_j^{\mathrm{T}} \dot{r}_i - r_i^{\mathrm{T}} \dot{r}_j \\
&= -k \sum_{l=1}^{m} a_{il} (r_j^{\mathrm{T}} r_l - (r_i^{\mathrm{T}} r_l) r_j^{\mathrm{T}} r_i) \\
&\quad -k \sum_{l=1}^{m} a_{jl} (r_i^{\mathrm{T}} r_l - (r_j^{\mathrm{T}} r_l) r_i^{\mathrm{T}} r_j) \\
&= k \sum_{l=1}^{m} a_{il} e_{lj} - k \left(\sum_{l=1}^{m} a_{il} \right) e_{ij} \\
&\quad -k \sum_{l=1}^{m} a_{il} e_{li} + k \left(\sum_{l=1}^{m} a_{il} e_{il} \right) e_{ij} \\
&\quad +k \sum_{l=1}^{m} a_{jl} e_{li} - k \left(\sum_{l=1}^{m} a_{jl} \right) e_{ij} \\
&\quad -k \sum_{l=1}^{m} a_{jl} e_{lj} + k \left(\sum_{l=1}^{m} a_{jl} e_{jl} \right) e_{ij}. \qquad (16)
\end{aligned}$$

Let $E = (e_{ij}) \in \mathbf{R}^{m \times m}$, $\alpha_i(E) = \sum_{l=1}^{m} a_{il} e_{il}$,

$$\alpha(E) = (\alpha_1(E), \alpha_2(E), \cdots, \alpha_m(E))^{\mathrm{T}} \in \mathbf{R}^m$$

and

$$\Lambda(E) = \begin{bmatrix} \alpha_1(E) & & & \\ & \alpha_2(E) & & \\ & & \ddots & \\ & & & \alpha_m(E) \end{bmatrix}.$$

Then we can rewrite (16) into the Riccati matrix differential equation

$$\dot{E} = -kLE - kEL^{\mathrm{T}} - k\alpha(E)\mathbf{1}^{\mathrm{T}} - k\mathbf{1}\alpha^{\mathrm{T}}(E) + k\Lambda(E)E + kE\Lambda(E), \quad (17)$$

where L is the Laplacian matrix of the topology.

In the following, we will use (17) to investigate whether E converges to zero exponentially. Before our main result on exponential synchronization, we first give some lemmas.

Lemma 4: Consider a sequence of unit vectors $r_i \in \mathbf{R}^n$ $(i = 1, 2, \cdots)$. Let $e_{ij} = 1 - r_i^{\mathrm{T}} r_j$ for any $i, j = 1, 2, \cdots$. Then

$$e_{ij} \le 2^s (e_{ik_1} + e_{k_1 k_2} + \cdots + e_{k_{s-1} k_s} + e_{k_s j}) \qquad (18)$$

for any s positive integers $k_1, k_2, \cdots, k_s$.

Proof: (By Mathematical Induction) For the case of $s = 1$, we consider the inequality

$$(2r_{k_1} - r_i - r_j)^T (2r_{k_1} - r_i - r_j) \ge 0. \qquad (19)$$

With simple calculations, it follows from (19) that

$$6 - 4r_{k_1}^{\mathrm{T}} r_i - 4r_{k_1}^{\mathrm{T}} r_j + 2r_i^{\mathrm{T}} r_j \ge 0, \qquad (20)$$

that is,

$$e_{ij} \le 2(e_{k_1 i} + e_{k_1 j}). \qquad (21)$$

Thus (18) holds for $s = 1$. Suppose that (18) holds for the case of $s - 1$. Then for the case of s, we have

$$\begin{aligned}
e_{ij} &\le 2(e_{ik_s} + e_{k_s j}) \\
&\le 2(2^{s-1}(e_{ik_1} + e_{k_1 k_2} + \cdots + e_{k_{s-1} k_s}) + e_{k_s j}) \\
&\le 2^s (e_{ik_1} + e_{k_1 k_2} + \cdots + e_{k_{s-1} k_s} + e_{k_s j}). \qquad (22)
\end{aligned}$$

■

Corollary 1: Assume $\mathscr{G}$ is a strongly connected digraph with weighted adjacency matrix $A = (a_{ij}) \in \mathbf{R}^{m \times m}$. Under the conditions of Lemma 4, the following statements hold:

(i) there exists a constant $c_1 > 0$ such that

$$e_{ij} \le c_1 \sum_{p=1}^{m} \sum_{q=1}^{m} a_{pq} e_{pq}, \ \forall \ i, j = 1, 2, \cdots, m; \qquad (23)$$

(ii) there exists a constant $c_2 > 0$ such that

$$\sum_{p=1}^{m} \sum_{q=1}^{m} a_{pq} e_{pq} \ge c_2 \sum_{i=1}^{m} \sum_{j=1}^{m} e_{ij}. \qquad (24)$$

Proof: (i) Since $\mathscr{G}$ is strongly connected, for any $i \ne j$, there is a directed path from j to i denoted by $ik_1k_2 \cdots k_s j$. Since all $a_{ik_1}, a_{k_1k_2}, \cdots, a_{k_s j}$ are positive, by (18), there is a constant $c_1 > 0$ such that

$$\begin{aligned}
e_{ij} &\le c_1 (a_{ik_1} e_{ik_1} + a_{k_1 k_2} e_{k_1 k_2} + \cdots + a_{k_s j} e_{k_s j}) \\
&\le c_1 \sum_{p=1}^{n} \sum_{q=1}^{n} a_{pq} e_{pq}. \qquad (25)
\end{aligned}$$

(ii) Let $c_2 = \frac{1}{c_1 m^2}$. Then Eq. (24) follows from (23). ■

Theorem 2: Consider Lohe model (4) with a strongly connected topology. Under the conditions of Theorem 1, after a finite time, the synchronization errors $e_{ij} = 1 - r_i^{\mathrm{T}} r_j$ $(i, j = 1, 2, \cdots, m)$ will converge to zero exponentially.

Proof: Since the topology is strongly connected, by algebraic graph theory, there is a unit vector $\beta = (\beta_1, \cdots, \beta_m) \in \mathbf{R}^m$ with each $\beta_i > 0$ and $\beta^T \mathbf{1} = 1$ such that $\beta^{\mathrm{T}} L = 0$. We construct a Lyapunov function for the dynamic equation (17) as follows:

$$V(E) = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \beta_i \beta_j e_{ij} = \frac{1}{2} \beta^{\mathrm{T}} E \beta.$$

Let

$$\Phi_\eta = \{E = (e_{ij}) \in \mathrm{R}^{m \times m} |\ 0 \le e_{ij} \le \eta, \forall i, j = 1, 2, \cdots\}$$

and

$$\Psi_\eta = \{E = (e_{ij}) \in \mathrm{R}^{m \times m} | V(E) < \beta_{\min}^2 \eta / 2,\ e_{ij} \ge 0, \forall i, j = 1, 2, \cdots\},$$

where $0 < \eta < 1$ and $\beta_{\min} = \min_{1\le i\le m}\beta_i$. It is easy to check that $\Psi_\eta \subset \Phi_\eta$. By Theorem 1, the synchronization is achieved. So after a finite amount of time, we have $E(t) \in \Psi_\eta \subset \Phi_\eta$. From $\beta^{\mathrm{T}}\mathbf{1} = 1$, $\beta^{\mathrm{T}}L = 0$ and (17), it follows that

$$\begin{aligned}
\dot{V}(E) &= \frac{1}{2}\beta^{\mathrm{T}}\dot{E}\beta \\
&= -k\beta^{\mathrm{T}}\alpha(E) + k\beta^{T}\Lambda(E)E\beta \\
&= -k\sum_{i=1}^{m}\beta_i\sum_{l=1}^{m}a_{il}e_{il} + k\sum_{i=1}^{m}\sum_{j=1}^{m}\beta_i\left(\sum_{l=1}^{m}a_{il}e_{il}\right)e_{ij}\beta_j \\
&= -k\sum_{i=1}^{m}\sum_{l=1}^{m}(1-\sum_{j=1}^{m}\beta_j e_{ij})\beta_i a_{il}e_{il} \\
&\le -k(1-\eta)\sum_{i=1}^{m}\sum_{l=1}^{m}\beta_i a_{il}e_{il} \\
&\le -k(1-\eta)\beta_{\min}\sum_{i=1}^{m}\sum_{l=1}^{m}a_{il}e_{il},
\end{aligned} \tag{26}$$

where $\beta_{\min} = \min\limits_{1\le i\le m}\beta_i$. Since the topology is strongly connected, by (26) and (24) of Corollary 1, we have

$$\begin{aligned}
\dot{V}(E) &\le -k(1-\eta)\beta_{\min}c_2\sum_{i=1}^{m}\sum_{l=1}^{m}e_{il}, \\
&\le -k(1-\eta)\frac{\beta_{\min}}{\beta_{\max}^2}c_2\sum_{i=1}^{m}\sum_{l=1}^{m}\beta_i\beta_l e_{il}, \\
&= -cV(E),
\end{aligned} \tag{27}$$

where $\beta_{\max} = \max\limits_{1\le i\le m}\beta_i$ and $c = k(1-\eta)\frac{\beta_{\min}}{2\beta_{\max}^2}c_2 > 0$. By the definition of Ψ_η and (26), we conclude that Ψ_η is a positively invariant set with respect to (17), and $V(E)$ converges to zero exponentially. Therefore, all the synchronization errors $e_{ij} = 1 - r_i^{\mathrm{T}}r_j$ $(i,j = 1,2,\cdots,m)$ tend to zero exponentially. ■

III. SIMULATION

In this section, we give some simulations to validate the obtained theoretical results.

Consider Lohe model (4) with $n = 3, m = 5$ and the adjacent matrix as follows:

$$A = \begin{bmatrix} 0\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 1\ 0 \\ 0\ 1\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0\ 1 \\ 0\ 1\ 0\ 0\ 0 \end{bmatrix}.$$

Then, the dynamical equations of Lohe model (4) with $k = 1$ are :

$$\begin{cases}
\dot{r}_1 = r_5 - (r_5^T r_1)r_1, \\
\dot{r}_2 = (r_1 - (r_1^T r_2)r_2) + (r_4 - (r_4^T r_2)r_2), \\
\dot{r}_3 = r_2 - (r_2^T r_3)r_3, \\
\dot{r}_4 = (r_3 - (r_3^T r_4)r_4) + (r_5^T - (r_5 r_4)r_4), \\
\dot{r}_5 = r_2 - (r_2^T r_5)r_5.
\end{cases}$$

Let the initial states of the Lohe oscillators be

$$\begin{aligned}
r_1(0) &= (-0.6545,\ 0.1391,\ 0.7431)^{\mathrm{T}}, \\
r_2(0) &= (-0.4973,\ -0.5523,\ 0.6691)^{\mathrm{T}}, \\
r_3(0) &= (0.6113,\ -0.6789,\ 0.4067)^{\mathrm{T}}, \\
r_4(0) &= (0.3025,\ 0.06425,\ 0.9511)^{\mathrm{T}}, \\
r_5(0) &= (0.6789,\ 0.3023,\ 0.6691)^{\mathrm{T}}.
\end{aligned}$$

Fig.1 shows that the time response curves of agent's are synchronized. Fig.2 shows the trajectories of all the agents of Lohe model converge to the same point on the unit sphere. In Fig.3, the exponential decay of the synchronization errors is displayed.

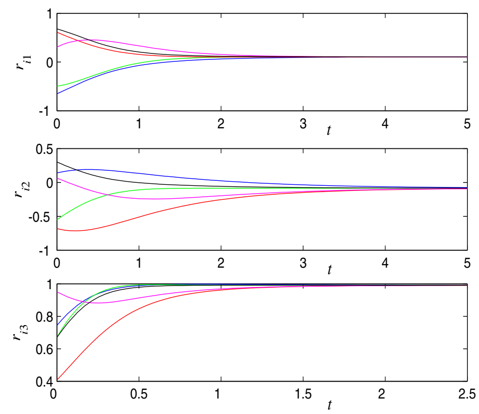

Fig. 1. Time response curves of of Lohe oscillators

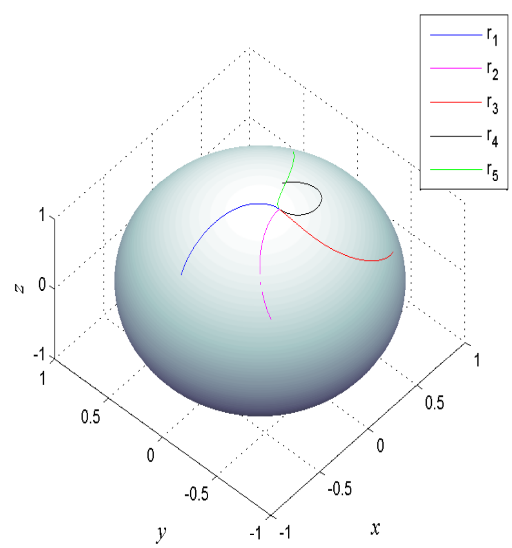

Fig. 2. Synchronization of Lohe model shown in the state space

IV. CONCLUSION

In this paper, the synchronization problem of Lohe oscillators with strongly connected topologies has been solved under the initial state limitations on a unit semi-sphere.

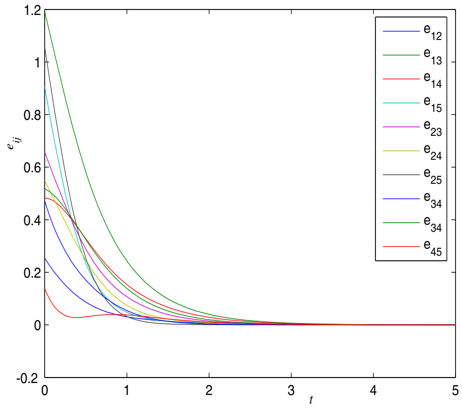

Fig. 3. Time response curves of the synchronization errors

A new form of matrix Riccati differential equation for synchronization errors has been proposed for the first time. The local exponential synchronization has been achieved for Lohe model with strongly connected digraph. Some numerical simulations have been given to illustrate the obtained theoretical results.

REFERENCES

[1] R. E. Mirollo and S. H. Strogatz, Synchronization of Pulse-coupled Biological Oscillators, *SIAM J. Appl. Math.*, vol 50, no. 6, pp. 1645-1662, 1990.
[2] D. Cumin, C. P. Unsworth, Generalising the kuramoto model for the study of neuronal synchronisation in the brain, *Phys. D*, vol. 226, no. 2, pp. 181-196, 2007
[3] B. Blasius, L. Stone, Chaos and phase synchronization in ecological systems, *Int. J. Bifurc. Chaos*, vol. 10, no. 10, pp. 2361-2380, 2000.
[4] S. Nair and N. E. Leonard, Stable Synchronization of Mechanical System Networks, *SIAM J. Control Optim.*, vol.47, no. 2 , pp. 661-683, 2008.
[5] C. Reynolds, Flocks,herds and schools: a disturbuted behavioral model, *Computer Graphics*, vol. 21, no. 4, pp. 25-34, 1987.
[6] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Sochet, Novel type of phase transitions in a system of self-driven particles,Phys.Rev.Lett., vol. 75, no. 6, pp. 1226-1229, 1995.
[7] Y. Kuramoto, Self-entrainment of a population of ciupled nonlinear oscillators, *Proceedings of International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics*, vol. 39, New York, 1975, pp. 420-422.
[8] D. Cumin and C. P. Unsworth, Generalising the Kuramoto model for the study of neuronal synchronisation in the brain, *Phys. D*, vol. 226, no. 2, pp. 181-196, 2007.
[9] F. Dörfler and F. Bullo, Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators, *SIAM J. Control Optim.*, vol. 50, no. 3, pp. 1616–1642, 2012.
[10] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, *Berlin: Springer-Verlag*, 1984.
[11] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente F. Ritort and R. Spigler, The Kuramoto model: a simple paradigm for synchronization phenomena, *Rev. Mod. Phys.*, vol. 77, no. 1 , pp. 137–185, 2005.
[12] S. H. Strogatz, From Kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators, *Phys. D*, vol. 143, no. 1-4, pp. 1–20, 2000.
[13] M. A. Lohe, Non-abelian Kuramoto model and synchronization, *J. Phys. A*, vol. 42, no. 39, pp. 395101, 2009.
[14] M. A. Lohe, Quantum synchronization over quantum networks, *J. Phys.*, vol. 43, no. 46, pp. 465301, 2010.
[15] R. Olfati-Saber, Swarms on sphere: A programmable swarm with synchronous behaviors like oscillator networks, *in IEEE Conf. on Decision and Control*, San Diego, 2006, pp. 5060–5066.
[16] J. Zhu, Synchronization of Kuramoto model in a high-dimensional linear space, *Phys. Lett. A*, vol. 377, no. 41, pp. 2939-2943, 2013.
[17] J. Zhu, High-dimensional Kuramoto model limited on smotth curved surfaces, *Phys. Lett. A*, vol. 378, no. 18-19, pp. 1269-1280, 2014.
[18] D. Chi, S. H. Choi, S. Y. Ha, Emergent behaviors of a holonomic particle system on a sphere, *J. Math. Phys.*, vol. 55, no. 5, pp. 052703, 2014.
[19] S. H. Choi, S. Y. Ha, Complete entrainment of Lohe oscillators under attractive and repulsive couplings, *SIAM J. Appl. Dyn. Syst.*, vol. 13, no. 4, pp. 1417-1441, 2014.
[20] C. Godsil and G. Royle, Algebraic graph theorem, *New York: Springer-Verlag*, 2001.
[21] G. Scutari, S. Barbarossa and L.Pescosolido, Distributed decision through self synchronizing sensor networks in the presence of propagation delays and asymmetric channels, *IEEE Trans. Signal Process.*, vol. 56, no. 4, pp. 1667-1684, 2008.
[22] Z. Lin, B. Francis and M. Maggiore, State agreement for continuous-time coupled nonlinear systems, *SIAM J. Control Optim.*, vol. 46, pp. 288-307, 2008.