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Port-Hamiltonian Systems on Graphs and Complexes

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Port-Hamiltonian systems theory has proven to be a versatile and insightful framework for the modeling, simulation and control of general (finite- and infinite-dimensional) multi-physics systems. Its main strengths are its compositionality (power-conserving interconnections of port-Hamiltonian systems are again port-Hamiltonian), as well as the fact that the port-Hamiltonian formulation makes explicit the presence of balance equations (i.e., conservation laws with source terms) implying the preservation (or decrease) of the internal energy, contrary to other network modeling approaches. Its most important feature is to define a geometric structure (called *Dirac structure*) capturing the network interconnection structure. The identification of these structural properties has turned out to be of crucial importance in the development of modular dynamical models for complex multi-domain physical systems, the numerical simulation preserving the physical invariants, and in the design of control strategies, which in the nonlinear case do not always aim at compensating nonlinearities, but instead make use of beneficial nonlinearities.

The aim of this mini-course is to present recent developments on the definition of Dirac structures and port-Hamiltonian systems associated with complex discrete topologies such as k-complexes (k-dimensional generalizations of graph topologies) and relate them with the port-Hamiltonian formulation of complex systems of balance equations defined on bounded spatial domains, their direct formulation on discrete spatial domains, and spatial semi-discretization using geometric structure preserving methods.

Main bibliographical references are [9, 5, 6, 8, 7, 4, 3, 1, 2, 10, 11]

The mini-course will consist of two blocks, each of 100 minutes.

Block 1: Port-Hamiltonian formulation of systems of balance equations

The theory of port-Hamiltonian systems is intimately linked to underlying systems of balance equations coupled by power continuous relations. This is already manifested in the modeling of classical examples like electrical networks and mass-spring-damper systems. From a mathematical point of view this is formalized by the geometric theory of finite-dimensional Dirac structures, which in the network case take a special form determined by the incidence structure of the underlying graph. These structures are naturally extended to the infinite-dimensional case, especially in the case of Stokes-Dirac structures as used in the modeling of systems of conservation laws defined on bounded spatial domains.

Talk 1.1: Finite-dimensional Dirac structures and graph Dirac structures

In this part we will recall the basic definition of finite-dimensional Dirac structures and their representations, as well as the definition of port-Hamiltonian systems defined with respect to these Dirac structures. It will be shown how systems of balance equations may be defined on directed graphs, together with their corresponding Dirac structures. This will be illustrated on the port-Hamiltonian formulation of classical physical network systems.

Talk 1.2: Stokes-Dirac structures and the modeling of systems of conservation laws

After a reminder of exterior differential calculus we will discuss Stokes-Dirac structures on smooth spatial domains with boundary. This will result in the definition of boundary port-Hamiltonian systems, including examples as the telegrapher's equations, beam models, the shallow water equations, and Maxwell's equations on bounded domains with power flow through the boundary. It will be shown how this formulation of reversible systems may be extended to irreversible physical systems, such as diffusive systems and how these systems may be formulated as port-Hamiltonian systems defined with respect to a Stokes-Dirac structure.

Talk 1.3: Advanced applications

In this part we will employ, and evaluate, the port-Hamiltonian theory in a number of applications, among which an experimental canal (making use of the port-Hamiltonian formulation of the shallow water equations), a sloshing system (coupled tank-beam system), a pressure swing adsorption process (a multilevel diffusion and adsorption process), an experimental device using a Ion polymer metal composites (including the model of a polymer electrolyte, its swelling and a beam model), and the Thermo-Magneto-Hydrodynamic model of the plasma of a Tokamak machine.

Block 2: Discrete formulation and semi-discretization of infinite-dimensional port-Hamiltonian systems

Both simulation and control by numerical methods of complex distributed parameter port-Hamiltonian systems – complex in the sense of complex geometries, multi-physics couplings, nonlinearities – require a discrete formulation or discretization which preserves the (geometric) power interconnection structure and the strict separation of the latter from the constitutive and dynamics equations.

The two talks of this second block of the mini-course are devoted to two different perspectives on the *numerical approximation* of port-Hamiltonian systems of conservation laws. We present recent results on the construction of port-Hamiltonian state space models for simulation and control under different boundary conditions and in arbitrary spatial dimension. We illustrate the discretization steps in a tutorial fashion on the examples of the 2D shallow water equations and the heat transfer in a metallic foam. We highlight the relations of the port-Hamiltonian approach – with its clear exposition of in- and output variables – to existing geometric discretization methods, from the cell method to discrete exterior calculus and finite element exterior calculus.

Talk 2.1: Discrete port-Hamiltonian formulation and numerical approximation of two conservation laws

We discuss the discrete formulation of systems of conservation laws in port-Hamiltonian form on dual chain complexes. Based on integral balance equations and purely topological information, this representation is exact and qualifies as a control model. The finite-dimensional approximation requires an energy discretization based on the constitutive equations and the geometry of the underlying staggered meshes. We recall the necessary notions from discrete exterior calculus, and illustrate them in terms of integration on discrete objects. By the construction of the primal and dual complex, one for each conservation law, and the localization of the different inputs on the system boundary, we construct port-Hamiltonian discrete models based on purely topological information. We present a finite volumes discretization of the constitutive equations and discuss the properties of the resulting structure-preserving discretization scheme.

Talk 2.2: Mixed Galerkin semi-discretization of port-Hamiltonian systems

Based on the weak formulation of the underlying Stokes-Dirac structure, we present the structure-preserving discretization of port-Hamiltonian systems of two conservation laws with a mixed Galerkin approach. In contrast to the method discussed in the first talk, the topological duality between the power variables is not considered a priori, which, for the finite element method, corresponds to the use of a single mesh for both conservation laws. The discrete bilinear form, which represents the power balance in the mixed geometric approximation bases has, however, a non-empty kernel. To obtain an approximate Dirac structure, linear maps on the discrete bond space must be defined, which preserve the power balance and remove the degeneracy of the discrete power product. The Dirac structure is completed by dynamics and a consistent approximation of the Hamiltonian in order to obtain finite-dimensional port-Hamiltonian state space models. The degrees of freedom in the power-preserving maps allow to parametrize the resulting family of semi-discretization schemes and to achieve "upwinding" in the directions of the boundary inputs. We illustrate the method on the example of Whitney finite elements on a 2D simplicial triangulation. Simulations and the eigenvalue analysis for linear systems prove the beneficial effects of different parametrizations according to the nature (hyperbolic vs. parabolic) of the considered system in the numerical solution of initial and boundary value problems.

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